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ANALYSIS AND DESIGN OF CRYSTAL OSCILLATORS

PART I

By

ERICH HAFNER

MAY 1964

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ANALYSIS AND DESIGN OF CRYSTAL OSCILLATORS

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ABSTRACT

The approach developed in this report appears to satisfy all the major requirements that must be placed on a unifying technique for oscillator analysis and design. As demonstrated on specific examples, the conditions for oscillation in a generally valid form can be processed to determine (a) the amplitude of oscillation in relation to the characteristics of the active device, (b) the requirements on the feedback network to operate the crystal unit according to its specifications, (c) the changes in frequency of oscillation in response to variations in any one of the circuit components and hence the values required for these components to obtain maximum stability, and (d) the output power in proportion to the power dissipated in the crystal unit.

The key to this approach lies in a graphical method for solution of the oscillator phase equation in the impedance plane. The impedance diagrams obtained thereby open the way to a thorough qualitative understanding of the cause-effect relationships in oscillator performance and provide the guidelines to bring the analytic expressions into a convenient form for quantitative work. Detailed discussions are carried out for the Pierce oscillator and the bridged-T oscillator to illustrate the practical application of the approach.

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ANALYSIS AND DESIGN OF CRYSTAL OSCILLATORS

1. INTRODUCTION

The literature on harmonic oscillators is very voluminous and for the most part is adequately referenced in textbooks on the subject.[1-4] However, in spite of the enormous number of details that have been dealt with, it is generally conceded that the successful design of an oscillator is still largely the result of very extensive laboratory experience.

This is believed to be due to the failure of the theoretical work reported so far to provide a conceptual framework which is readily assimilated by the serious student of the art and which explains qualitatively the interrelation of all the elements in oscillator circuits.

A rigorous theoretical treatment of the properties of a practical oscillator circuit is possible in principle only. The extraordinary complexity of the equations encountered prevents their solution and interpretation in the general case. The success of any analysis of oscillators therefore depends upon the judicious selection of the approximations that have to be made to keep the problem tractable and upon the proper formulation of the analytic expressions in such a manner that they are most susceptible to interpretation.

An oscillator is a physical system capable of one or more steady-state conditions that are oscillatory in nature. Its behavior can be accurately described by a nonlinear differential equation if its component values and the properties of the active element are known. In most practical cases this approach is limited to systems with only one or two degrees of freedom. The differential equation can then be solved graphically by Lienard's construction[5] or, if the static voltage current characteristic of the active element can be represented by a cubic polynominal, by analytical methods.[6]

Inherent in both procedures is the assumption that the operating point of the active element is independent of the state of oscillations. In reality this is hardly ever the case and any information obtained from these methods concerning the transient behavior of the model, as well as the steady-state amplitude of the oscillations, has to be modified suitably if this effect should be accounted for. Since the major significance of these nonlinear oscillator theories is in the qualitative description of the function of the active element during oscillations, the additional complications just referred to do not diminish their fundamental importance for the understanding of oscillators. Considering the limitations involved, no quantitative results can be expected from this approach unless the differential equations are solved with the aid of computers for specific situations.[7]

The qualitative information provided by the nonlinear theories, however, covers all aspects of the oscillatory behavior that can occur in a system with two or more energy storage elements. In particular, they identify the factors that act to limit the amplitude of oscillations and illustrate how the energy that is dissipated in the circuit is replenished through the active element. The waveform of the oscillator signal, i.e., its harmonic content, can be obtained by no other means except by the techniques of the nonlinear theories.

While the harmonic content is important for its influence on the fundamental frequency,[8] the nonlinear theories do not appear adaptable to the type of analysis required by the circuit designer. The purpose of any design technique is to provide guidelines for the synthesis of a passive network which will assure that first of all the behavior of the fundamental component

of the output signal meets the design objectives. Hence, the properties of the passive network have to be analyzed and to do so it becomes necessary to approximate the nonlinear element in the circuit by an equivalent linear element that faithfully reflects the action of the former in regard to the fundamental component of the signal. This then makes it possible to treat the oscillator with the tools of linear network theory. Obviously, this approach will no longer yield any information concerning the harmonic components of the signal directly.

The nonlinear theories provide the conceptual basis for the equivalent linearization of the nonlinear active element, in particular they illustrate that the values of the equivalent "linear" parameters must depend upon the amplitude of the signal. The fundamental component of the solution to the nonlinear differential equation can be shown[9] to satisfy a linear differential equation whose damping term vanishes once the steady-state amplitude has been reached. If no transit time effects are involved, this damping term contains the parameters of the static current vs control voltage characteristic of the active nonlinear element. From it the characteristic of the equivalent linear element can be determined, in principle, as a function of amplitude.

This approach has not always been followed consistently in the past and the appreciation in the literature of the factors responsible for amplitude limiting within the context of the equivalent linearization is frequently less than accurate.

There are notable exceptions, however. Plots of the effective ac current through the active element vs the effective ac control voltage for a given set of operating conditions and for a given frequency have been used very successfully for the discussion and evaluation of oscillator and regenerative amplifier behavior.[10] A plot of this type is known as Moeller's "Schwinglinie"[11] and replaces the static current vs control voltage curve in characterizing the active nonlinear element. Though of impressive utility, this technique and its implications have frequently been ignored in the post World War II literature.

In recent years, however, another technique has been devised. Rather than plotting the effective ac current vs the effective ac control voltage, Reich[12] chose to use the g_m vs effective control voltage curves to qualitatively illustrate several aspects of oscillator behavior. Since the transconductance g_m is the ratio of ac current to ac control voltage for the fundamental frequency signal, the Reich diagram is essentially equivalent to the Schwinglinien diagram and the discussions of oscillator performance can readily be translated from one to the other. For oscillator design work, however, the Reich diagram is superior because the effective g_m is the quantity that actually appears in the analytic expressions. In addition, since the g_m is one of the small-signal parameters appearing in the four-terminal-network representation of a vacuum tube, the concept is quite easily extended to other active elements, such as transistors.

With the properties of the active element adequately understood and its nonlinear ac characteristics represented by appropriate "constants," it becomes possible to turn the attention to the investigation of the passive network. Two areas are of interest: First, the network should be such as to assure that oscillations occur, and occur with a predetermined signal amplitude. Second, the frequency of oscillation must be as stable as possible and the effects of variations in any one of the network parameters on frequency and amplitude should be known.

The major part of this report deals with these problems. In spite of numerous attempts, it has in the past not been possible to develop a unifying concept that could readily be applied to examine the cause-effect relationships in a large group of oscillator circuits under a variety of conditions. Major cause of the difficulties is the extraordinary complexity encountered when treating a practical oscillator network by the techniques customarily employed. The approximations required to arrive at usable expressions reduce the actual network invariably

to a skeleton model which makes it no longer possible to recognize its qualitative features on a broader basis. As a consequence, some rather basic properties of harmonic oscillators, in particular of those employing quartz crystals are only partially understood.

2. THE BASIC OSCILLATOR EQUATIONS

An idealized oscillator circuit is shown in the upper part of Fig. 1. The general impedances Z_1, Z_2 and Z_3 form the feedback network and the electron tube supplies the energy necessary to sustain oscillations. If the concept of the equivalent linearization is used, the vacuum tube can be treated as though it were a linear element whose ac output current i_1 is proportional to the ac input voltage v_g and 180° out of phase with it

$$i_1 = -g_m v_g \quad . \tag{1}$$

Since the tube, however, is actually a nonlinear element, the value of g_m depends upon the amplitude of the signal v_g. We will assume that g_m as a function of A, the amplitude of the fundamental component of the oscillator signal, gives a steadily decreasing curve such as shown in the lower part of Fig. 1. This assumption meets with the most commonly encountered behavior and does not impose a serious restriction on the developments of this report. The value of g_{mo}, which represents the transconductance of the tube for infinitely small signals, as well as the exact shape of the g_m vs amplitude curve depends upon the type of tube and the dc potentials on all the tube electrodes. However, this curve is independent from the impedances Z_1, Z_2, and Z_3 in the feedback network except for the effect of these impedances on the dc electrode potentials. Additional details concerning the effective transconductance of a vacuum tube will be found in Appendix A.

The voltage v_g in Fig. 1 is related to the input current i_1 by the relation

$$v_g = i_1 \frac{Z_1 Z_2}{Z_s} \quad . \tag{2}$$

with

$$Z_s = Z_1 + Z_2 + Z_3 \quad . \tag{2a}$$

The relation (2), because of (1), reduces to the condition for oscillations in the form

$$g_m \frac{Z_1 Z_2}{Z_s} = -1 \quad . \tag{3}$$

If we set

$$Z_i = |Z_i| e^{j\theta_i} \quad (i = 1, 2, 3, s) \tag{4}$$

and separate the real from the imaginary part in (3), we obtain the two conditions

$$\theta_1 + \theta_2 - \theta_s = \pi \tag{5}$$

$$\frac{|Z_s|}{|Z_1||Z_2|} = g_m < g_{mo} \tag{6}$$

which have to be satisfied simultaneously for steady-state oscillations to occur. As further explained in Appendix B, the ratio of the impedance magnitudes must be smaller than g_{mo}. The value of this ratio determines the g_m which must be exhibited by the active device during steady-state oscillations; and hence, it determines the amplitude of the oscillations by way of the amplitude dependence of g_m shown in the lower part of Fig. 1. If large amplitude oscillations are required, it is necessary to make the difference $(g_{mo} - g_m)$ as large as possible. To obtain small amplitudes the opposite is true, of course.

It might be well to point out in this connection that an incidental change in any one of the impedance magnitudes will change the value of the impedance ratio in (6) and hence cause g_m to change. Since, however, a variation in g_m can take place only if the amplitude of oscillations changes, the incidental variation in one of the impedance magnitudes will cause a change in amplitude which can be quite appreciable if the slope of the g_m vs amplitude curve is small. In order to steepen the slope of this curve, particularly for low amplitude operation, one or more of the dc bias potentials are frequently made a function of amplitude through the use of AGC circuits.

While Eq. (6) determines then the amplitude of oscillation, Eq. (5) determines the frequency at which the oscillations take place.

Since all three impedances Z_1, Z_2 and Z_3 in Fig. 1 and therefore Z_s as well are, in general, functions of frequency, one or more of the phase angles in (5) will also be functions of frequency and the equation can be satisfied only at one or, more often, at several discrete points. If there are several, stable oscillations will occur at that root of (5) at which (6) requires the lowest value of g_m.

Any analytic method to solve (5) is impractical for all but the most idealized circuits. However, it is possible and indeed most instructive to solve this equation without any additional approximations graphically in the impedance plane.

While this technique is by no means restricted to quartz crystal oscillators, we will use in the following the Pierce and Miller circuits to illustrate the approach.

Before we go into it, however, we believe it quite useful to review in some detail the impedance diagram of a crystal unit[13] because of its fundamental importance for crystal oscillator operation and performance.

3. THE IMPEDANCE DIAGRAM OF A QUARTZ CRYSTAL UNIT

The familiar equivalent electrical circuit of a crystal unit with load capacitor is shown on the upper right-hand side of Fig. 2 and next to it, it is drawn again to identify the symbols which will be used in describing its properties. Only the narrow frequency range of a particular crystal response is of interest and it will be assumed that X_1 the reactance of the motional arm is the only quantity that changes appreciably with frequency in this range, i.e., X_o, X_L and R_1 are assumed to be constants.

If the impedance $Z_x = R + jX$ of a crystal unit is measured point by point as a function of frequency, it will be found to describe a circle in the R-X plane. Such an impedance diagram of a crystal unit is shown as the heavily drawn circle in Fig. 2. It can be described by the equation

$$\left(R - \frac{X_o^2}{2R_1}\right)^2 + (X - X_o - X_L)^2 = \left(\frac{X_o^2}{2R_1}\right)^2 . \tag{7}$$

The vector $Z_x = |Z_x|e^{j\theta_x}$ representing the impedance between the two terminals of the crystal network follows this circle in a clockwise direction as the frequency of operation is changed upward through the range of the crystal response. Each point of this circle corresponds to a different frequency and a measure of the change in Z_x with frequency can be obtained by providing the circle with a frequency calibration.

This is the purpose of the set of circles drawn in Fig. 2 with thin lines. Each one of these circles, which follow the equation

$$R^2 + \left(X - X_o - X_L + \frac{X_o^{\,2}}{2\,(X_o + X_1)}\right)^2 = \left(\frac{X_o^{\,2}}{2\,(X_o + X_1)}\right)^2, \qquad (8)$$

belongs to a different value for X_1 and hence to a different frequency. Those drawn into the diagram, Fig. 2, are separated from one another by an equal increment in X_1, and hence by an equal number of cycles, over a certain range of X_1 around $X_1 = 0$. Since the intercepts of these circles with the impedance circle of the crystal unit identify the frequencies at which Z_x assumes the corresponding values, a diagram such as shown in Fig. 2 provides a rather illustrative picture of the behavior of the crystal impedance throughout the response range.

As an example, we will consider the effects of a variation in the load impedance X_L. From Eq. (7) and (8), it can be seen that any change in X_L effects a simple translation of the entire diagram along the imaginary axis. If X_L and X_o are both capacitive as has been assumed here, an increase in the magnitude of X_L will cause the entire set of circles to move downward. Consequently, since the frequency calibration on the impedance circle is not affected, the frequency at which Z_x has a given phase angle, say $\theta_x = 0$, will increase. Z_x will thereby move into a range of the impedance circle where the frequency calibration becomes more open; and in an oscillator that should be frequency modulated, a given amount of modulation in X_L will cause a correspondingly smaller modulation of the output frequency. On the other hand, if X_L is made inductive, for example, by using a condenser and coil in series in place of C_L, the distance between the center of the impedance circle and the real axis can be made quite small. Since now the resonance frequency of the network, i.e., the frequency at which $\theta_x = 0$, is in a range of the impedance circle where the frequency calibration points are closer together, the same amount of modulation in X_L will cause a much larger modulation of the output frequency.

To help avoid a trivial trap in the use of the impedance diagram later on, it should be emphasized that the impedance vector Z_x starts, of course, at the origin of the R-X coordinate system, while the impedance circle touches the imaginary axis at $X = X_o + X_L$.

4. THE PIERCE OSCILLATOR

In an oscillator such as shown in Fig. 1, the crystal network can take the place of any one of the three impedances Z_1, Z_2 and Z_3. If it is Z_3 and the impedances Z_1 and Z_2 are capacitive, the resulting configuration is that of the well-known Pierce oscillator, shown schematically in Fig. 3.

For the moment, it will be assumed that the impedances Z_1 and Z_2 have been chosen initially and that it shall be determined if this configuration will oscillate with a particular crystal unit and at what point on the impedance circle the crystal will be operated. According to the condition (5), oscillations can take place only if θ_s, the phase angle of the sum vector $Z_s = Z_1 + Z_2 + Z_3$, is given by

$$\theta_s = \theta_1 + \theta_2 + \pi \,.$$

We can plot, therefore, the impedances Z_1 and Z_2 in the impedance plane and find the angle $(\theta_1 + \theta_2)$ as well as $\theta_s = \theta_1 + \theta_2 + 180°$, such as shown in the left-hand diagram of Fig. 3. Hence, for the phase relation to be satisfied, the sum vector Z_s must fall along the broken line in the first quadrant.

To find the magnitude of Z_s, it is only necessary to first obtain $Z_1 + Z_2$ and to use this impedance as the origin for the impedance diagram of the crystal network. This is shown in the right-hand diagram of Fig. 3. The broken line circle denotes all possible values for Z_3, while Z_s can only assume values along the line θ_s = const. The intercept of these two lines determines Z_s and Z_3. Apparently Z_3, the impedance of the crystal network, must always have an inductive reactance that is larger than the sum of the capacitive reactances of Z_1 and Z_2 unless the resistive components of the latter are zero, i.e., unless θ_1 and θ_2 are both $-90°$.

If the diagram has been drawn to scale, the length of Z_s can be measured and the ratio g_m

$$\frac{|Z_s|}{|Z_1||Z_2|} = g_m$$

determined. If g_m is smaller than g_{mo}, oscillations will take place and, from the g_m vs amplitude curve, the amplitude of these oscillations can be found. Their frequency follows from the value of Z_3, together with Eq. (3). The latter, incidentally, will generally be found much easier to use, either in the form shown or solved for X_1, than most other relations for this purpose.

From the diagram in Fig. 3, it can be readily appreciated that, for example, a decrease in $X_o + X_L$ will cause the impedance circle to intercept the line θ_s = const. at a lower point. The magnitude of Z_s decreases and with it the ratio g_m, calling for higher amplitude oscillation. This effect, of course, becomes more pronounced as the curvature of the crystal impedance circle increases either due to a larger value of R_1 or a smaller X_o. Drawing a set of constant frequency circles (8) into the diagram of Fig. 3 provides, furthermore, a very direct impression of the interdependence of frequency of oscillation, crystal resistance R_1 and load reactance X_L. It is well to take note of the fact that the constant frequency circles (8) are independent of R_1.

5. THE MILLER OSCILLATOR

If the crystal network takes the place of Z_2 in Fig. 1, we obtain the basic diagram of the Miller oscillator as shown in Fig. 4. In order to explain the qualitative features of this oscillator, we will assume now that Z_1 and Z_3 have been chosen initially and that we wish to determine the impedance, Z_2 in this case, which the crystal network is required to exhibit during steady-state oscillations, provided oscillations are possible.

The phase relation (5) requires now

$$\theta_s - \theta_2 = \pi + \theta_1 \ . \tag{9}$$

Since, however, θ_2 and θ_s are, of course, not independent from one another, the problem of solving (5) is now slightly more complicated, though not essentially different from the previous case. In Section 4 we had to determine, at first, the loci of all values Z_s for which (5) is satisfied. This was done in the left-hand diagram in Fig. 3 and the curve happened to be the straight line θ_s = const. In the right-hand diagram of this Figure, we then found the actual

operating point as the intercept of this curve with the one describing those values of Z_3 that the crystal network is capable of assuming.

An identical procedure can be followed to solve (5) if Z_2 is the unknown impedance. Disregarding for the moment the physical significance of the θ's, solving Eq. (9) becomes a purely geometric problem. The solution describes a circle in the impedance plane. Part of this circle is drawn into the left-hand diagram in Fig. 4. It goes through the origin of the impedance plane and through the end point of the vector $Z_1 + Z_3$. A third point on this circle is found most conveniently by assuming $\theta_2 = 90°$. According to (5), θ_s is then given by $\theta_s = \theta_1 - \frac{\pi}{2}$ which is readily obtained from the graph. The circle is then drawn through these three points, however, only the solid portion of it corresponds to physically realizable values of Z_2 and hence Z_s. In the right-hand picture of Fig. 4, the impedance diagram of the crystal unit is shown superimposed on the graph just obtained, with its origin at $Z_1 + Z_3$. Again the intercepts of these two curves identify those values of Z_s and Z_2 that satisfy the phase relation and at the same time are permitted for the crystal network. Of the two intercepts, only the upper one is of interest because it corresponds to a substantially lower value of g_m. If the latter is smaller than g_{mo}, the circuit will oscillate at the frequency at which the crystal exhibits the required impedance Z_2.

The diagram in Fig. 4 shows quite clearly that even though the impedance Z_2 is inductive, the crystal unit is still operated in its low resistive region. By no means should the fact that the Miller oscillator is known as a parallel resonance oscillator be taken to imply that the crystal unit is operated near its antiresonance point, i.e., in the high resistance range of its impedance circle. If the crystal is operated correctly, that is, according to specifications, the Pierce oscillator, as well as the Miller oscillator, will have the same output frequency, which is only a different way of stating that the crystal is operated on the same point of its impedance circle in both cases.

Considerations such as were made for the Pierce oscillator concerning the effects of a change in series load reactance X_L can of course easily be translated to apply for the Miller oscillator. With slightly more effort, the graphical construction in Fig. 4, together with (7) and (8), can be used to illustrate the effects of a change in X_o, such as introduced by a load element in parallel with the static capacitance of the crystal unit. X_L may or may not be left to go to zero.

The graphical constructions in Fig. 3 were based on the assumption that the terminating impedances Z_1 and Z_2 are known, whereas those in Fig. 4 require knowledge of Z_1 and Z_3 or Z_2 and Z_3. In general, therefore, regardless of where the crystal network is, the procedure followed in Fig. 3 has to be used to determine Z_3 while that of Fig. 4 has to be used to determine one of the terminating impedances.

6. THE STABILITY RELATIONS

Principally for the purpose of demonstrating some of the features of crystal operation in an oscillator, we have assumed in Sections 4 and 5 that it is the operating point of the crystal unit that has to be determined while the other impedances in the circuit are known. However, one problem in designing an oscillator is frequently to assure, by proper choice of the other circuit components, that the crystal unit is operated at its specified operating point in order to obtain frequency correlation. This means that the impedances Z_1 and Z_2 in the case of the Pierce oscillator or Z_1 and Z_3 for the Miller oscillator have to be determined, together with the proper values of X_L and X_o, to obtain the desired operation.

The conditions for oscillation can obviously be met with a very wide range of circuit impedances. In fact, it is well known that only the most rudimentary precautions have to be taken to obtain a configuration that will oscillate somehow. The decisive question that has to be answered is, "What are the optimum values of the circuit impedances for a given application?"

For frequency control applications, the most pertinent criterion for the performance is undoubtedly the degree to which the desired frequency of oscillation is maintained. It can easily be appreciated from either Fig. 3 or Fig. 4 that any change in, for example, Z_1, will require either Z_2 or Z_3 or both to change if the conditions for oscillations should again be satisfied for the new value of Z_1, i.e., for $Z_1 + \Delta Z_1$.

The relationship between the various ΔZ's can be found quite generally by differentiating the condition for oscillation (3). We find,

$$\frac{\Delta Z_s}{Z_s} - \frac{\Delta Z_1}{Z_1} - \frac{\Delta Z_2}{Z_2} = -\frac{\Delta_{gm}}{g_m} \cdot \tag{10}$$

Because the Z's are generally functions of a number of parameters such as resistors, condensers, and inductors, as well as of frequency, i.e.,

$$Z_1 = Z_1\ (\alpha_{11}, \alpha_{12}, \alpha_{13}, \ldots ; \omega), \tag{11}$$

the differentials are

$$\Delta Z_1 = \sum_k \frac{\partial Z_1}{\partial \alpha_{1k}} \Delta \alpha_{1k} + \frac{\partial Z_1}{\partial \omega} \Delta \omega \tag{12}$$

with like expressions for Z_2, Z_3 and ΔZ_2, ΔZ_3, respectively.

Each one of the Z's has, as a complex quantity, a magnitude and a phase angle

$$\Delta Z = |\Delta Z| e^{j\theta_{\Delta Z}} \ . \tag{13}$$

If (4) and (13) are substituted into (10), the equation can be separated into real and imaginary parts. The real part of (10) determines the changes in g_m and hence in the amplitude of oscillation. The imaginary part, however, contains the equation for $\Delta\omega$, the change in frequency. With $\Delta Z_s = \Delta Z_1 + \Delta Z_2 + \Delta Z_3$, which follows from (2a), the imaginary part of (10) becomes

$$\frac{|\Delta Z_3|}{|Z_s|} \sin(\theta_{\Delta Z_3} - \theta_s) + \frac{|\Delta Z_1|}{|Z_s|} \sin(\theta_{\Delta Z_1} - \theta_s) - \frac{|\Delta Z_1|}{|Z_1|} \sin(\theta_{\Delta Z_1} - \theta_1) + \\ + \frac{|\Delta Z_2|}{|Z_s|} \sin(\theta_{\Delta Z_2} - \theta_s) - \frac{|\Delta Z_2|}{|Z_2|} \sin(\theta_{\Delta Z_2} - \theta_2) = 0 \ . \tag{14}$$

With (12) and (13), it is possible to compute from (14) the frequency shift $\Delta\omega/\omega$ caused by a given set of parameter changes ($\Delta\alpha$). While the equation looks formidable in its general form, it frequently reduces to quite manageable expressions when a particular circuit is being

considered.

Using the diagrams of Fig. 3 or Fig. 4, it will be realized that each one of the terms in (14) represents a phase angle change. For example, the second term is the ratio of that component of ΔZ_1 which is normal to Z_s to the magnitude of Z_s and hence, represents the change in θ_s due to the action of ΔZ_1. Similarly, the third term is the change in θ_1 due to ΔZ_1 a.s.o. The information contained in (14) could have been derived in principle by appropriately differentiating (5); however, the resulting equation would be of very limited use and would again have to be transformed into the form (14) for practical application. Nevertheless, the mere possibility is of great value conceptually in interpreting the relation (14). For example, it demonstrates quite clearly that the sum of all phase angle changes caused by ΔZ_1 and ΔZ_2 must be compensated by an equal and opposite change in θ_s due to the action of a ΔZ_3 of appropriate magnitude. The change in θ_3, the phase angle of the impedance Z_3, does not enter the stability relation directly.

An important aspect of (14) is the general type of design information that can be extracted from it. We can use again the Pierce oscillator to illustrate this point.

In the case of the Pierce oscillator, Z_3 represents the crystal network and $\partial Z_3/\partial\omega$ will ordinarily be so large compared to $\partial Z_2/\partial\omega$ and $\partial Z_1/\partial\omega$ that the latter two can be neglected compared to the first. ΔZ_1 and ΔZ_2 therefore represent changes in Z_1 and Z_2 that are caused by variations in any one or more of the parameters α_{1k} and α_{2k} of (11). Since in this type oscillator, Z_1 and Z_2 are nearly always the impedances of networks consisting of several elements in parallel, it is more convenient to replace ΔZ_1 by $|Z_1|^2|\Delta Y_1|$ and ΔZ_2 by $|Z_2|^2|\Delta Y_2|$. If, in addition, it is assumed that the parameters of the crystal network, α_{3k}, remain constant, Eq. (14) becomes

$$\frac{\Delta\omega}{|Z_s|}\left|\frac{\partial Z_s}{\partial\omega}\right| \sin(\theta_{\Delta Z_3} - \theta_s) +$$

$$+\frac{|Z_1|^2}{|Z_s|}\,|\Delta Y_1|\,\sin(\theta_{\Delta Z_1} - \theta_s) - |Z_1|\,|\Delta Y_1|\,\sin(\theta_{\Delta Z_1} - \theta_1) + \tag{15}$$

$$+\frac{|Z_2|^2}{|Z_s|}\,|\Delta Y_2|\,\sin(\theta_{\Delta Z_2} - \theta_s) - |Z_2|\,|\Delta Y_2|\,\sin(\theta_{\Delta Z_2} - \theta_2) = 0\,.$$

The first term in (15) can be transformed further by using an expression for the quality factor of the crystal network that is valid over the entire range of the crystal response. Such a formula for the crystal Q can be shown to be given by

$$Q_o = \frac{\omega}{2\,\mathrm{Real}\,Z_3}\left|\frac{\partial Z_3}{\partial\omega}\right|\,. \tag{16}$$

With (16) and (6), Eq. (15) can be written in the following form:

$$-\frac{2\Delta\omega}{\omega} Q_{eff} =$$

$$= \frac{1}{gm} \frac{|Z_1|}{|Z_2|} |\Delta Y_1| \sin(\theta_{\Delta Z_1} - \theta_s) - |Z_1||\Delta Y_1| \sin(\theta_{\Delta Z_1} - \theta_1) + \tag{17}$$

$$+ \frac{1}{gm} \frac{|Z_2|}{|Z_1|} |\Delta Y_2| \sin(\theta_{\Delta Z_2} - \theta_s) - |Z_2||\Delta Y_2| \sin(\theta_{\Delta Z_2} - \theta_2)$$

whereby

$$Q_{eff} = Q_o \frac{\text{Real } Z_3}{|Z_s|} \sin(\theta_{\Delta Z_3} - \theta_s) \cdot \tag{18}$$

Evidently, a given set of $|\Delta Y_1|$ and $|\Delta Y_2|$ will cause the smallest change in frequency if Q_{eff} and g_m are as large as possible and $|Z_1|$ and $|Z_2|$ are small.

Q_{eff} is the effective quality factor of the crystal network in the oscillator circuit. It always is smaller than Q_o, the quality factor of the crystal network alone, with the factors contributing to the degradation apparent from (18). A more extensive discussion of the effective quality factor will be given in Section 8, where it will be shown how the expression (18) can be transformed to yield the information required in the design of a feedback network with minimum Q degradation.

Z_1, Z_2, and Z_3 are related to one another and the effective transconductance of the tube by (6). The lower limit for $|Z_1|$ and $|Z_2|$ is set by the available g_{mo} of the active device. To obtain the highest frequency stability, therefore, requires that g_m is as close to g_{mo} as possible. This means by implication that the amplitude of the oscillations will be small unless the g_m vs amplitude curve is extremely flat. The latter, of course, is undesirable because of poor amplitude stability as pointed out in Section 2. As the crystal frequency is sensitive to amplitude variations, there is an optimum point of operation on the g_m vs amplitude curve that will give the highest frequency stability and this applies also if the shape of this curve is modified by AGC action. Particularly in connection with AGC, it is important to note that the frequency deviation $\Delta\omega/\omega$ according to (17) depends actually upon the impedance magnitudes and only because of (6) on the effective g_m of the active device. AGC should be applied sparingly so as not to reduce the operating point of the active device and hence the available g_m more than necessary.

It has been assumed so far that the parameters of the crystal network α_{3k} remain constant while the terminating impedances Z_1 and Z_2 are subject to change. According to (17), it is principally the effective quality factor of the crystal unit that determines the sensitivity of the oscillator frequency to the variations in Z_1 and Z_2. An essentially different type of condition, however, exists if $\Delta Z_1 = \Delta Z_2 = 0$ is assumed and the parameters of the crystal network are varied. In this case Eq. (14) reduces to the equation

$$\theta_{\Delta Z_3} - \theta_s = 0 \tag{19}$$

which states that Z_3 can only change along the line θ_s = const. This latter fact can easily be verified by using the diagram in Fig. 3. Since Z_1 and Z_2 are assumed constant, θ_s is not

affected by a variation in any one of the parameters α_{3k} and hence remains constant. A specific example of the situation existing now has already been considered in Section 3 where the effects of a variation in X_L were discussed. It can be shown that the change in frequency of oscillations in response to a variation of the parameters α_{3k} is controlled primarily by the capacitance ratio C_o/C_1 of the crystal unit and is independent to a large measure of the quality factor Q_o.

We find, therefore, two generally valid criteria for designing an oscillator for maximum frequency stability. First of all, it is necessary to reduce to a minimum all those factors which tend to degrade the Q of the crystal unit once it is incorporated into the circuit, i.e., Q_{eff} as defined by (18) must be made as large as possible.

The second criterion derives from the fact that the frequency changes caused by a change in any one of the components in Z_3 are controlled by the C_o/C_1 ratio of the crystal network. A large ratio is desired if the resulting frequency change should be small. For optimum stability, the C_o/C_1 ratio should be kept as large as practical. To a large extent this criterion applies to the proper choice of the crystal unit to be used in the circuit, since generally, for a given Q_o, the crystal with the higher resonance resistance will have the larger C_o/C_1 ratio.

All the considerations made so far for the Pierce oscillator apply, with some modifications, to the Miller oscillator as well. However, in practical applications, it will be found that the latter is more difficult to design properly and, in addition, its impedance level appears to be restricted to a rather unfavorable range.

7. GENERALIZATION OF THE OSCILLATOR EQUATIONS

When dealing with vacuum-tube oscillators, particularly those employing pentodes, the approach taken in Section 2 might be found acceptable for most engineering applications without much hesitation. The nature of presently available transistors, however, leaves no choice but to examine the problem more carefully in order to determine how much of the foregoing analysis needs to be modified and amended for it to apply to transistor oscillators as well.

A perfectly general representation of a harmonic oscillator is shown in Fig. 5. It consists of two four-terminal networks in a cyclic arrangement. Network I shall contain the active element, or elements if more than one is used, while Network II contains the ac feedback loop. The load can be considered a part of Network II.

Within the concept of equivalent linearization, it is possible, using matrix theory, to completely characterize the small-signal performance of Network I by a set of four constants, the most appropriate for the configuration being the a parameters. The input-output relations for Network I are given by

$$\begin{aligned} v_1 &= a_{11} v_2 - a_{12} i_2 \\ i_1 &= a_{21} v_2 - a_{22} i_2 \, . \end{aligned} \tag{20}$$

It is important to emphasize that the parameters (a_{jk}) are entirely independent from the properties of Network II. Their values do, of course, depend upon the biasing conditions and upon the signal level in a manner similar to that explained in Section 2 for g_m.

The input-output relations for Network II are

$$v_3 = a'_{11} v_4 - a'_{12} i_4$$

$$i_3 = a_{21} v_4 - a'_{22} i_4 \quad . \tag{21}$$

Because of the cyclic arrangement of the networks shown in Fig. 5, $i_3 = -i_2$, $v_3 = v_2$, $i_1 = -i_4$ and $v_4 = v_1$ must hold during steady-state oscillations. It can be verified quite readily with (20) and (21) that for these relations to be possible, it is required that

$$1 - a_{21} a'_{12} - a_{12} a'_{21} - a_{22} a'_{22} - a_{11} a'_{11} + \Delta^a \Delta^{a'} = 0 \tag{22}$$

with

$$\Delta^a = a_{11} a_{22} - a_{12} a_{21} \quad ; \quad \Delta^{a'} = a'_{11} a'_{22} - a'_{12} a'_{21} \quad .$$

Equation (22) expresses the conditions for oscillation in general form, subject only to the limitations imposed by the concept of equivalent linearization. It applies to vacuum tube and transistor oscillators alike.

If a feedback loop in the form of a π network is assumed, as suggested in Fig. 5, the parameters (a'_{ik}) are given by

$$(a'_{ik}) = \begin{pmatrix} 1 + \dfrac{Z_3}{Z'_2} & Z_3 \\ \dfrac{Z'_s}{Z'_1 \, Z'_2} & 1 + \dfrac{Z_3}{Z'_1} \end{pmatrix} ; \quad \begin{aligned} \Delta^{a'} &= 1 \\ Z'_s &= Z'_1 + Z'_2 + Z_3 \quad . \end{aligned} \tag{23}$$

While every linear passive network can, in principle, be represented in this form and hence the generality of the following considerations is not restricted by this assumption, in practical applications it will be found convenient to base the analysis on the actual form of the feedback network and to use the corresponding expressions for the a' parameters. In this sense the following discussions apply directly to all oscillators with only three essential nodes in the feedback network and serve as guidelines for the analysis of oscillators with more complicated networks.

The a parameters of a transistor are not widely used and it will be more convenient to express the equations in terms of the common emitter "h" parameters. The latter are less abstract in their meaning and are usually obtained by direct measurement. Again, the fact that the common-emitter parameters are chosen here is only a matter of convention and does not impose any limitations on the validity of the equations. The two four-terminal networks in Fig. 5 can always be defined such as to conform to this convention.

The well-known relations between the a and h parameters are given by

$$(a_{ik}) = \begin{pmatrix} -\dfrac{\Delta^h}{h_{21}} & -\dfrac{h_{11}}{h_{21}} \\ -\dfrac{h_{22}}{h_{21}} & -\dfrac{1}{h_{21}} \end{pmatrix}; \quad \begin{aligned} \Delta^a &= -\frac{h_{12}}{h_{21}} \\ \Delta^h &= h_{11}h_{22} - h_{12}h_{21} \,. \end{aligned} \tag{24}$$

If (23) and (24) are inserted into (22), it will be found possible, after some re-arrangement of terms but without approximations, to write the general condition for oscillations in the form

$$g_m \frac{Z_1 Z_2}{Z_s} = -1 \tag{25}$$

and thus to reduce it to the basic equation discussed in the foregoing sections of this report. The symbols in (25) are defined as follows:

$$g_m = \frac{h_{21}}{h_{11}}\left(1 - \frac{h_{12}}{h_{21}} + \frac{h_{12}}{h_{11}} Z_3\right) \tag{26}$$

$$Z_s = Z_1 + Z_2 + Z_3 \tag{27}$$

$$\frac{1}{Z_1} = \frac{1}{Z_1'} + \frac{\Delta^h}{h_{11}} \tag{28}$$

$$\frac{1}{Z_2} = \frac{1}{Z_2'} + \frac{1}{h_{11}} \,. \tag{29}$$

It is observed that h_{11}/Δ^h in (28) is the impedance seen looking into the output of Network I with its input short-circuited and that h_{11} in (29) is the impedance seen looking into the input of Network I with its output short-circuited. Since no approximations have been made in the derivation of (25) other than those implied by the equivalent linearization, Eq. (25) applies equally well if Network I represents any type of amplifier, no matter how simple or complicated its structure.

To apply the methods of oscillator analysis developed for an ideal current generator to any actual circuit configuration, it is only necessary to consider the short-circuit input impedance as well as the short-circuit output impedance of the amplifier as part of the feedback network and to define the strength i_2 of the equivalent ideal current generator as $-i_2 = g_m v_1$ with the form of g_m stipulated by (26).

The parameters of the active device are still separated from those of the feedback network in the expressions (28) and (29). The advantage of dividing the oscillator into an active and a passive network as shown in Fig. 5 is still retained therefore; namely, that the parameters of each one can be measured independently from the other. Since there are no basic restrictions on the manner in which the oscillator is divided into the two networks, this division can be carried out so as to assure that the parameter measurements can be made in a convenient and meaningful fashion. Frequently, this will require that part or all of the biasing network be included in the active network.

According to the foregoing, then, the active four-terminal network in an oscillator can be represented by a simple equivalent circuit as indicated in Fig. 6. Within the concept of the equivalent linearization, this representation is exact if g_m is defined by (26).

The impedances h_{11} and h_{11}/Δ^h in Fig. 6 depend upon the bias conditions and upon the signal levels within the active network. In oscillators with minimum Q degradation, however, the impedances Z'_2 and Z'_1 will be small compared to h_{11} and h_{11}/Δ^h and knowledge of the values of the latter for extremely small signals will be adequate in most practical cases. Their variation with signal level will frequently be negligible in calculating the parameters of the feedback network. For the most general case, the direct measurement of h_{11} as a function of bias condition and signal level does not present any undue difficulties, but a similar evaluation of h_{11}/Δ^h will not always be possible. Its value at other than extremely small-signal levels will have to be approximated.

The effective transconductance g_m as given by (26) is also a function of the bias conditions and of the signal levels within the active network. In most cases of practical significance, it will be possible to approximate g_m by h_{21}/h_{11} which can be evaluated directly. Since only forward transmission is involved here, the signal levels within the active device are defined if v_1 and v_2 are defined. Very frequently the dependence of h_{21}/h_{11} on v_2 is so slight in the range of interest as to be negligible and curves of h_{21}/h_{11} as a function of the amplitude of v_1 with the bias conditions at zero amplitude as parameters are adequate for most design purposes. When necessary, such curves can readily be extended to include v_2 or some appropriate function of v_2 as an additional parameter.

More substantial difficulties, however, arise if $g_m = h_{21}/h_{11}$ is not a valid approximation. To include the term with Z_3 in (26) requires that an iteration process be used in order to solve (25) by the graphical techniques explained previously. Consideration of the amplitude dependence of h_{12}/h_{21} and h_{12}/h_{11} is only possible in a very approximate manner.

The amplitude and phase relations in the general case are obtained from (25) by separating the real and imaginary parts just as (5) and (6) were found from (3). They are

$$\theta_s = \theta_1 + \theta_2 + \theta_g + \pi \tag{30}$$

$$\frac{|Z_s|}{|Z_1||Z_2|} = |g_m| < |g_{mo}| \tag{31}$$

whereby

$$g_m = |g_m| e^{j\theta_g} \quad . \tag{32}$$

Though the existence of a finite phase angle θ_g does require some minor modification, the graphical method explained previously is readily applied to the solution of (30). A finite θ_g simply means that Z_s the sum vector falls no longer on the line $\theta_1 + \theta_2 + \pi$ = const. as in Fig. 3, rather it must fall on the line $\theta_1 + \theta_2 + \pi + \theta_g$ = const. The effects of this modification are most clearly demonstrated qualitatively by using the corresponding diagrams. For example, since θ_g is usually negative, i.e., of the same sign as θ_1 and θ_2 in a Pierce oscillator, it is noted that a θ_g of proper value can cause θ_s to become zero even if $\theta_1 + \theta_2$ is less than 180°. Under certain conditions, therefore, it is possible that a finite θ_g will result in improved oscillator performance. To identify these conditions, however becomes increasingly more difficult as the number of variables increases and their relative tendency to to change in value with time and environmental conditions is considered.

8. THE GENERALIZED STABILITY RELATIONS

The fact that g_m in (25) is no longer a real quantity in the general case also causes some modifications of the stability relations derived previously in Section 6.

Since (25) is of the same form as (3), differentiating (25) leads again to the relation (10) between the changes in the quantities Z_1, Z_2, Z_3 and g_m, with only the definition of the symbols altered according to the considerations of the preceding Section.

The imaginary part of (10) is now given by

$$\frac{|\Delta Z_3|}{|Z_s|}\sin(\theta_{\Delta Z_3}-\theta_s)+$$
$$+\frac{|\Delta Z_1|}{|Z_s|}\sin(\theta_{\Delta Z_1}-\theta_s)-\frac{|\Delta Z_1|}{|Z_1|}\sin(\theta_{\Delta Z_1}-\theta_1)+ \tag{33}$$
$$+\frac{|\Delta Z_2|}{|Z_s|}\sin(\theta_{\Delta Z_2}-\theta_s)-\frac{|\Delta Z_2|}{|Z_2|}\sin(\theta_{\Delta Z_2}-\theta_2)=\frac{|\Delta g_m|}{|g_m|}\sin(\theta_{\Delta g}-\theta_g)\ .$$

Evidently this equation is formally identical to the stability relation (14) for the idealized oscillator, save for the fact that (33) contains the change in the phase angle θ_g caused by a given $\theta_{\Delta g}$. Of course, θ_s now depends upon θ_g according to (30) and g_m, Z_1 and Z_2 depend upon the parameters of the active device according to (26), (28) and (29). Except for these differences, which certainly must be kept in mind, all the factors considered during the discussion and interpretation of (14) apply equally well to the stability relation (33) for a generalized oscillator.

Of particular importance is the fact that the expression (18) for the effective quality factor remains unchanged with only the symbols redefined and that the two criteria established in Section 6 for the design of an oscillator for optimum frequency stability are valid also in the general case.

The expression corresponding to (17) now takes the form

$$-\frac{2\Delta\omega}{\omega}Q_{eff}=\frac{1}{|g_m|}\frac{|Z_1|}{|Z_2|}|\Delta Y_1|\sin(\theta_{\Delta Z_1}-\theta_s)-|Z_1||\Delta Y_1|\sin(\theta_{\Delta Z_1}-\theta_1)+$$
$$+\frac{1}{|g_m|}\frac{|Z_2|}{|Z_1|}|\Delta Y_2|\sin(\theta_{\Delta Z_2}-\theta_s)-|Z_2||\Delta Y_2|\sin(\theta_{\Delta Z_2}-\theta_2)- \tag{34}$$
$$-\frac{|\Delta g_m|}{|g_m|}\sin(\theta_{\Delta g}-\theta_g)\ .$$

9. THE EFFECTIVE QUALITY FACTOR

Because of the fundamental importance of the effective quality factor for the proper operation of a crystal oscillator, it is quite necessary to bring the expression (18) into a form which

can be used more advantageously in the design of a feedback network with the highest Q_{eff} possible. We will use the following definitions and relations:

$$Z_1 = R_1 + jX_1 = \frac{G_1}{G_1^2 + B_1^2} - j\frac{B_1}{G_1^2 + B_1^2} = G_1|Z_1|^2 - jB_1|Z_1|^2$$

$$Z_2 = R_2 + jX_2 = \frac{G_2}{G_2^2 + B_2^2} - j\frac{B_2}{G_2^2 + B_2^2} = G_2|Z_2|^2 - jB_2|Z_2|^2 \tag{35}$$

$$Z_3 = R_3 + jX_3 \quad ; \qquad \text{Real } Z_3 = R_3 \tag{36}$$

$$|Z_s| \cos\theta_s = R_1 + R_2 + R_3 \ .$$

Because of (36), which follows most readily from a diagram such as shown in Fig. 3, one can write (18) in the form

$$Q_{eff} = Q_o \frac{\sin(\theta_{\Delta Z_3} - \theta_s)\cos\theta_s}{1 + \dfrac{R_1 + R_2}{R_3}} \tag{37}$$

which still does not contain any approximations. With (35) and (31), one can obtain the equally general expression

$$\frac{R_1 + R_2}{R_3} = \frac{1}{|g_m|}\frac{|Z_s|}{R_3}\left(G_1\frac{|Z_1|}{|Z_2|} + G_2\frac{|Z_2|}{|Z_1|}\right) \tag{38}$$

which can be approximated by

$$\frac{R_1 + R_2}{R_3} \cong \frac{1}{|g_m|}\left(G_1\frac{|Z_1|}{|Z_2|} + G_2\frac{|Z_2|}{|Z_1|}\right), \tag{39}$$

if the major part of $|Z_s|$ is the resistance R_3 of the crystal network. When (39) is used in (37), this approximation will cause only second-order errors in the following expression for Q_{eff}

$$Q_{eff} = Q_o \frac{\sin(\theta_{\Delta Z_3} - \theta_s)\cos\theta_s}{1 + \dfrac{1}{|g_m|}\left(G_1\dfrac{|Z_1|}{|Z_2|} + G_2\dfrac{|Z_2|}{|Z_1|}\right)} \ . \tag{40}$$

Further transformations do not appear advisable in the general case. If G_1, G_2, R_3 and g_m are given quantities, the optimum ratio $|Z_1|/|Z_2|$ can be determined from (40), however, the process is quite cumbersome since it is usually not possible to ignore the dependence of the numerator on θ_1 and θ_2 according to (30). The form (40) appears to be quite suitable for the use of an iteration process, carried out analytically or graphically in the impedance plane.

A substantially more convenient expression for the effective quality factor can be found, however, if, as is very often the case, it can be proved justified to assume

$$\theta_{\Delta Z_3} \doteq \pm 90° \quad ; \quad \theta_g \doteq 0 \quad . \tag{41}$$

The numerator in (40) then reduces to

$$\cos \theta_s \sin (\theta_{\Delta Z_3} - \theta_s) \doteq \cos^2 \theta_s \tag{42}$$

which can be approximated for reasonably small angles by

$$\cos^2 \theta_s \approx 1 - 2\theta_s^2 = 1 - 2\left(\theta_1 + \frac{\pi}{2} + \theta_2 + \frac{\pi}{2}\right)^2 \approx$$

$$\approx 1 - 2\left(G_1 |Z_1| + G_2 |Z_2|\right)^2 \quad . \tag{43}$$

Because of (31):

$$\left(G_1|Z_1| + G_2 |Z_2|\right)^2 \approx \frac{R_3}{g_m}\left(G_1^2 \frac{|Z_1|}{|Z_2|} + G_2^2 \frac{|Z_2|}{|Z_1|} + 2G_1G_2\right) \tag{44}$$

so that the expression (40) can be rewritten approximately in the highly useful form:

$$D = \frac{Q_o - Q_{eff}}{Q_o} = \frac{|Z_1|}{|Z_2|}\frac{G_1}{g_m}(1 + 2R_3G_1) + \frac{|Z_2|}{|Z_1|}\frac{G_2}{g_m}(1 + 2R_3G_2) + \frac{4G_1G_2R_3}{g_m} \quad . \tag{45}$$

Except for terms that are small of higher order, the relation (45) is valid if the assumptions (41) are met and the phase angle θ_s is small enough for (43) to hold.

Most frequently the values of G_1, G_2, g_m and R_3 are already determined once an active device and a crystal unit have been selected, leaving the ratio $|Z_1|/|Z_2|$ in (45) to be adjusted for minimum degradation. Evidently this requires

$$\frac{|Z_1|^2}{|Z_2|^2} = \frac{G_2}{G_1}\frac{1 + 2R_3G_2}{1 + 2R_3G_1} \quad . \tag{46}$$

The minimum value for (45) is then given by

$$D_{min} = \left(\frac{Q_o - Q_{eff}}{Q_o}\right)_{min.} = \frac{2}{g_m}\sqrt{G_1 G_2 (1 + 2R_3 G_1)(1 + 2R_3G_2)} + \frac{4G_1G_2R_3}{g_m} \quad . \tag{47}$$

Among the interesting features of (47) is its dependence on R_3, the resistive component of the crystal network. As R_3 is decreased from some large value, the Q degradation becomes eventually independent of R_3 and the use of a crystal unit with a still lower R_3 will not result in an improved effective quality factor. Moreover, a crystal unit with a very low resistance has a correspondingly low C_o/C_1 ratio and its use would violate one of the general design criteria

established before; namely, that C_o/C_1 should be as large as practical. In a sense, the value of D_{min} for $R_3 = 0$ can be considered a figure of merit of the active device.

Evidently, there is an optimum value for R_3 and even without going into a detailed analysis of the complete stability relation, it can be assumed that in most cases R_3 should be just large enough so as not to contribute appreciably to the Q degradation. Usually this requirement can be met with a fairly wide range of R_3 values.

A specific example will illuminate this point further. Using a 2N700 transistor, one might find $G_1 = 10^{-4}$ mho, $G_2 = 10^{-2}$ mho, $g_m = 4 \times 10^{-2}$ mho. The minimum Q degradation $D_{min}(R_3)$ as a function of R_3 can then be found from (47), e.g., $D_{min}(0) = 5\%$, $D_{min}(50) = 7\%$, $D_{min}(100) = 8.5\%$, $D_{min}(150) = 11.5\%$. Hence, there is evidently little to be gained from using an R_3 below 50Ω with this particular set of values, and even an R_3 as high as 150Ω might still be considered acceptable in some cases.

The conditions for oscillation (5) and (6), or, in the general case (30) and (31) specify the magnitude of the product $|Z_1||Z_2|$ while the requirement for minimum Q degradation leads to a value for the ratio $|Z_1|/|Z_2|$. Together, therefore, these relations are sufficient to determine the impedances Z_1 and Z_2 required by an oscillator with optimum frequency stability. Since in a large number of applications, it will not be necessary to solve the general expression (40) for the optimum ratio $|Z_1|/|Z_2|$, the relation (46) becomes the most important result of this section. It permits the rapid evaluation of the optimum ratio $|Z_1|/|Z_2|$ which can then be used in (31) to find $|Z_1|$ and $|Z_2|$.

If the resulting Q degradation is low enough, the phase angles θ_1 and θ_2 will frequently be close enough to $\pi/2$ to permit the approximations $|Z_1| \doteq |X_1|$ and $|Z_2| \doteq |X_2|$ so that the conditions (31) and (46) can be further simplified to

$$\frac{|X_1|^2}{|X_2|^2} \doteq \frac{G_2(1 + 2R_3G_2)}{G_1(1 + 2R_3G_1)} \tag{48}$$

$$g_m \doteq \frac{R_3}{|X_1||X_2|} . \tag{49}$$

With X_1 and X_2 determined from (48) and (49), it only remains to choose X_L in Z_3 such that $X_1 + X_2 + X_L$ equals the reactance of the load capacitance specified for the crystal unit to be used and the design of the oscillator is essentially completed.

One factor, however, has not yet been considered explicitly; namely, the power dissipation in the various components.

10. DRIVE LEVEL AND OUTPUT POWER

During steady-state oscillations, the voltages and currents in an oscillator circuit have all to be related to one another in a very definite manner, and hence their relative magnitudes are firmly established. In fact it is precisely this relationship that leads to the conditions for oscillations, no matter how they are formulated. The absolute magnitude of the voltages and currents remain undetermined unless the amplitude dependence of the nonlinear elements is known. The left side of Eq. (22), for example, is a function of the transistor parameters (a_{ik}), all of which depend upon the amplitude of the signal. Normally there will only be one set of values (a_{ik}), assumed at a definite value for the signal amplitude, for which the right side of (22) is zero; for any other amplitude, it will be larger or smaller than zero and the conditions

for steady-state oscillations are not satisfied. An accurate calculation of this critical amplitude is possible in principle if the parameters (a'_{ik}) of the passive network are known and the parameters (a_{ik}) of the active device are available as functions of signal amplitude. Such calculations would obviously be quite complicated and, because of the inherent sources of error, are very likely not justified even in extreme cases.

A more practical approach can be taken if the phase angles of the impedances Z_1 and Z_2 as defined by (28) and (29) are in the order of or less than $10°$. It is then possible to approximately consider Z_1 and Z_2 as independent of amplitude, using nominal values for h_{11} and Δh or values measured in an impedance bridge at approximately the desired amplitude if increased accuracy is required. This leaves g_m or $|g_m|$, respectively, as the only amplitude dependent quantity in (25) and hence in (31). g_m as defined by (26) can most frequently be approximated by h_{21}/h_{11}.

Using reasonable care, it is readily possible to adjust the parameters of the feedback network such that the impedance ratio $|Z_s|/|Z_1|\cdot|Z_2|$ falls within 10% of a predetermined value. If, therefore, a curve such as indicated in Fig. 1 has been found experimentally and the g_m corresponding to the desired amplitude of oscillations determined, the amplitude actually obtained can be in error by an amount that clearly depends on the slope of this curve. Particularly at low amplitudes, a 10% error in the ordinate of this curve will generally be found intolerable and the desired amplitude has to be obtained by manual or automatic fine adjustments of the bias conditions of the active device or by adjustments in the feedback network.

It was stated before in Section 7 that the active network in an oscillator is most appropriately characterized by the g_m vs input voltage curves with the bias conditions as parameters. Hence, the input voltage to the active device assumes a special role, and all other voltages and currents in the oscillator are best referred to it, using the conditions for oscillation whenever necessary. These conditions are clearly not required to compute the rf power dissipated in the elements of the passive network.

For the following considerations in this section, we will again use the basic circuit diagram shown in Fig. 1, keeping in mind, however, that the impedances Z_1 and Z_2 now include, respectively, the effective output and input impedance of the active device. We will assume also that the harmonic content of the oscillator signal is negligible. If the rms value of the input voltage $\bar{v}_g$ is known, the power P_k dissipated in the impedance Z_k ($k = 1, 2, 3$) can be computed to

$$\begin{aligned} P_1 &= \frac{|Z_2 + Z_3|}{|Z_2|^2} G_1 \bar{v}_g^{\,2} \\ P_2 &= G_2 \bar{v}_g^{\,2} \\ P_3 &= \frac{R_3}{|Z_2|^2} \bar{v}_g^{\,2} \,. \end{aligned} \tag{50}$$

The effective value $\bar{v}_1$ of the voltage across Z_1 is given by

$$\bar{v}_1^{\,2} = \frac{|Z_2 + Z_3|^2}{|Z_2|^2} \bar{v}_g^{\,2} \,. \tag{51}$$

Normally the load will be part of Z_1 and the power delivered into the load can be computed from P_1 for any particular case. For the present purpose it will suffice to consider P_1 as synonymous with P_L, the power into the load. If the crystal network is Z_3 and the only resistive component in this network is due to the crystal unit, P_3 will be the power dissipated in the crystal.

Since the ratio of the power delivered to the load to the power dissipated in the crystal unit is frequently of considerable concern, we will derive now an expression for P_1/P_3 that very clearly illustrates the influence of the various circuit parameters on this ratio. From (50)

$$\frac{P_1}{P_3} = \frac{G_1}{R_3} |Z_2 + Z_3|^2 \; . \tag{52}$$

In the general case, the magnitude of the vector $Z_2 + Z_3$ is best obtained from diagrams such as used for the graphical analysis of the phase relations (30), which of course will be very similar to the ones shown in Fig. 3 or Fig. 4. If, however, the phase angle of Z_2 is close to $90°$ and the phase angle of the sum vector Z_s close to zero, we can set approximately

$$Z_2 + Z_3 \approx R_3 + j(X_2 + X_3) \approx R_3 - jX_1 \tag{53}$$

since, under these conditions $X_1 + X_2 + X_3 \doteq 0$. With (53) and (49), which also holds under these conditions, the power ratio (52) can be written as

$$\frac{P_1}{P_3} = G_1\left(R_3 + \frac{1}{g_m} \frac{|X_1|}{|X_2|} \right) . \tag{54}$$

In a well-designed oscillator, the Q degradation should be kept at a minimum, that is, the ratio $|X_1|/|X_2|$ is given by (48). Hence,

$$\frac{P_1}{P_3} = G_1 R_3 + \frac{1}{g_m}\sqrt{G_1 G_2 \frac{1 + 2 R_3 G_2}{1 + 2 R_3 G_1}} \tag{55}$$

is the expression for the power ratio in a Pierce-type oscillator whose components are adjusted for a relative maximum in effective Q. G_1 and G_2 are the conductances of the terminating networks Z_1 and Z_2, respectively, and R_3 is the resistive component of the crystal network.

Using again the approximations that lead from (52) to (55), one obtains in a similar manner the following expressions:

$$\frac{P_2}{P_3} = \frac{G_2}{g_m} \frac{|X_2|}{|X_1|} = \frac{1}{g_m} \sqrt{G_1 G_2 \frac{1 + 2 R_3 G_1}{1 + 2 R_3 G_2}} \tag{56}$$

$$P_3 = \bar{v}_g^{\,2} g_m \frac{|X_1|}{|X_2|} = \bar{v}_g^{\,2} g_m \sqrt{\frac{G_2 (1 + 2 R_3 G_2)}{G_1 (1 + 2 R_3 G_1)}} \tag{57}$$

$$\bar{v}_1^{\,2} = P_3 \left(R_3 + \frac{1}{g_m} \frac{|X_1|}{|X_2|}\right) = \bar{v}_g^{\,2} g_m R_3 \sqrt{\frac{G_2 (1 + 2 R_3 G_2)}{G_1 (1 + 2 R_3 G_1)}} + \bar{v}_g^{\,2} \frac{G_2(1 + 2 R_3 G_2)}{G_1(1 + 2 R_3 G_1)} \; . \tag{58}$$

It will be noted that even if the major part of the conductance G_1 is the load, the output power from a Pierce-type oscillator is only a small fraction of the power dissipated in the crystal unit and any attempt to increase this fraction will result in increased Q degradation, even under optimum conditions. Since, however, the Q degradation can be made extremely small, the Pierce-type oscillator is ideally suited for applications where maximum stability is required. The relations (55) through (58), together with (47), will be found useful in a given situation to reach an acceptable compromise between output power and Q degradation. Other type oscillators, such as the bridged T oscillator discussed in Appendix C, have to be used when high output power is of primary importance and the lower stability can be tolerated.

11. THE DESIGN OF THE FEEDBACK NETWORK

The detailed equations developed in the preceding sections are based on a feedback circuit in the form of a π network. The expressions derived apply most directly to oscillators whose feedback network is physically arranged in this form with the conventional Pierce oscillator as the simplest representative of this class. As an illustration, we will now go through some of the considerations that have to be made in the design of a Pierce oscillator.

We will assume that we are given a specific transistor and a specific crystal unit and are supposed to design an oscillator of maximum frequency stability.

To get a first orientation, one will start with the simplest equations available, using manufacturers' published data for the parameters of the transistor and the crystal unit to obtain a rough approximation for the element values in the feedback network and the corresponding signal levels. From this one can determine if the use of the simple equations is justified, i.e., if the assumptions made in their derivation hold in the case considered, or if the more complicated relations will have to be dealt with. Once a typical g_m vs amplitude curve is available, it can also be determined if the signal level is high enough for the nonlinearities in the transistor to be adequate for stabilizing the amplitude at the desired value, or if it will be necessary to use AGC or additional nonlinear elements in the circuit.

For example, a 2N700 transistor operated at 2 ma in the frequency range around 5 Mc typically has the following parameters: $h_{11} = 100\Omega$, $h_{21} = 4$. A 5-Mc fifth overtone crystal unit typically has a resonance resistance of 130Ω. If the capacitors and inductors to be used in the feedback network are assumed to be essentially lossless, we find from (29) and (35) that $G_2 = 1/h_{11} = 10^{-2}$ mho. G_1 will be determined principally by the power delivered to the output amplifier. We will assume $G_1 = 10^{-4}$ mho. If we set $g_m = h_{21}/h_{11}$ as a first approximation to (26) and if we assume this quantity to be real, we have all the information necessary to compute from the condition for minimum Q degradation (48) the ratio $|X_1|/|X_2|$ and from the condition for oscillation (49) the product $|X_1| \cdot |X_2|$ and hence to determine $|X_1|$ and $|X_2|$. We find $|X_1| = 246\Omega$, $|X_2| = 13.2\Omega$. The resulting Q degradation is 11% according to (47).

From the relations (54) or (55) through (58), we find: $P_1/P_3 \doteq 6 \times 10^{-2}$, $P_2/P_3 \doteq 1.3 \times 10^{-2}$, $P_3 \doteq .75\,\bar{v}_g^{\,2}$, $\bar{v}_1 \doteq 21\bar{v}_g$. Hence, if the drive level of the crystal unit should be $P_3 \doteq 10^{-6}$ watts, it follows that $P_1 \doteq 6 \times 10^{-8}$ watts, $P_2 \doteq 1.3 \times 10^{-8}$ watts, $\bar{v}_g = 1.15 \times 10^{-3}$ volts, and $\bar{v}_1 \doteq 2.4 \times 10^{-2}$ volts. At these low voltage levels, the transistor, by itself, is very nearly a linear device. Although it is possible to achieve steady-state oscillations under these conditions even without AGC, merely by careful manual adjustment of the bias conditions, the amplitude stability is generally not adequate. We have to conclude that the oscillator will require some form of artificial level control if the crystal power P_3 should not exceed 1 microwatt.

Regardless of how this level control is accomplished, it will bring about a modification of the active device parameters, which has to be considered. In most cases, AGC will principally affect the g_m vs amplitude dependence, leaving G_1 and G_2 unaltered. The absolute value of g_m for the desired signal amplitude can usually be adjusted over a wide range by adjusting the operating conditions of the active device and, as stated previously, should be adjusted to as high a value as practical.

For the present purpose we will assume that the values of G_1, G_2 and g_m are not appreciably altered by the level control mechanism and continue to regard the values for $|X_1|$ and $|X_2|$ found above as valid first-order approximations.

After these preliminary steps, it is now necessary to determine if the circuit does satisfy the conditions for which the approximate relations are valid. The relation (48) can be used whenever the conditions (41) are met and when θ_s is small enough for the expansion (43) to hold.

$\theta_{\Delta Z_3}$ in (41) is the angle of the tangent to the impedance circle of the crystal unit at the point of operation (see Fig. 2 and 3) and it will be close to $\pm 90°$ if $X_0 + X_L$ is very much smaller than the diameter of the impedance circle X_o^2/R_1. If necessary, a parallel coil has to be used to resonate with C_o at the operating frequency to assure that $\theta_{\Delta Z_3}$ is indeed nearly $\pm 90°$ so that the first condition in (41) will rarely ever limit the validity of the simplified equation (48).

The second condition in (41) pertains to the active device and the extent to which it is met has to be determined by actual measurements. It has to be kept in mind that g_m is defined by (26) and that the approximation $g_m = h_{21}/h_{11}$ is valid only if the correction terms involving h_{12} are negligible. For the 2N700 transistor at 5 Mc and 2 ma, these correction terms cause an error of less than one percent and the phase angle θ_g is less than $3°$. The second condition (41), therefore, can also be considered satisfied in the present example.

The error in the expansion (43) is about 2.5% if $\theta_s = 10°$ and increases to 15% for $\theta_s = 20°$. With $G_1 = 10^{-4}$ and $|X_1| = 246\Omega$ as found before, $\theta_1 \doteq \pm 88.5°$ while $G_2 = 10^{-2}$ and $X_2 = 13.2\Omega$ give $\theta_2 \doteq \pm 82.5°$ so that $\theta_s = \theta_1 + \theta_2 - \pi = \pm 9°$, well within the range of values for which (43) is acceptable. In the present example, Eq. (48) can be used without serious error in place of (40), or (18) in the most general case, to determine the ratio of $|Z_1|/|Z_2|$ required for minimum Q degradation.

In a like manner it can be shown that only negligible errors are incurred in the present case if (49) is used instead of (31) as the condition for oscillation.

With the validity of the approximate relations established, it is obvious that better approximations to $|X_1|$ and $|X_2|$ can be obtained if the parameters of the particular active device and the particular crystal unit on hand are used in the expressions (48) and (49). This may be found necessary or desirable for high precision applications even if fine adjustments are often made on the oscillator itself.

So-called parallel resonance crystal units, i.e., crystal units intended to operate into a capacitive load, such as in a Pierce oscillator, are adjusted during manufacture such that their impedance circle, (see Fig. 2) crosses the real axis at their nominal frequency if the load capacitor C_L is $32\,\mu\mu f$. This value of C_L, though not significant by itself, is laid down by international convention to provide a universal reference for frequency correlation between crystal manufacturer and crystal user. There is one point on the impedance circle, therefore, where the crystal has to be operated to exhibit the correct frequency. Because of the frequency dependence upon drivelevel, the crystal will exhibit this frequency, even if operated at the correct point on the impedance circle, only if the power dissipated in

the crystal has the value specified for it.

From the diagram in Fig. 3, it can be seen that in a Pierce oscillator with given values of Z_1 and Z_2, the crystal is operated at that point of its impedance circle at which it crosses the line θ_s = const. To obtain frequency correlation, it is necessary that

$$X_1 + X_2 + X_L - |Z_s| \sin \theta_s = X\,(32\mu\mu f) \tag{59}$$

and that the power dissipated in the crystal has the specified value. The load capacitor C_L in series with the crystal has to be adjusted to satisfy this condition.

The relation (59) identifies, incidentally, that quantity, namely, $|Z_s| \sin \theta_s$ that is responsible for the miscorrelation in frequency observed when crystal units with a wide range of resistances are substituted in an oscillator, i.e., crystal units whose impedance circles have widely different diameters. Evidently, if Z_1, Z_2 and X_L should remain constant in this case, frequency correlation can be maintained exactly only if either θ_s is zero or if the crystals are modified, for example, by a series resistor to always give the same $|Z_s|$. In an oscillator designed for optimum stability, θ_s will be as small as possible and a certain amount of variation in $|Z_s|$ may frequently be found tolerable, provided the concomitant variation in crystal drivelevel and output power is acceptable also.

While compromises may be required in the general case, it will evidently always be possible to determine those values of $|X_1|$, $|X_2|$ and X_L, or more generally Z_1, Z_2 and X_L for which the sacrifice in performance is at a minimum. In doing so, the graphical solutions of the phase relation, such as illustrated in Fig. 3 or Fig. 4, will be found a very valuable tool in visualizing the cause-effect relationships between the various impedances involved, while the actual calculations of the numerical values can nearly always be carried out analytically.

It will be noted that at no point in the entire analysis so far has it been necessary to specify how the impedances Z_1 and Z_2 determined in this manner are to be realized physically. In fact it has not even been necessary to specify whether X_1 and X_2 are to be capacitive or inductive reactances; it is only required that θ_1 and θ_2 are either both positive or both negative.

A number of factors have to be considered in determining how the impedances Z_1 and Z_2 thus specified are to be realized physically. First, they have to provide a dc path for biasing and one of them, generally Z_1, has to include the loading effect due to the amplifier stage following the oscillator. Second, their frequency dependence must be such as to prevent oscillations at any but the desired frequency. It is this latter requirement that ordinarily rules out the possibility that θ_1 and θ_2 be both positive in crystal oscillators.

The preferred way of biasing a transistor is through a large resistance in the emitter path and very small resistances in the base and collector paths.[14,15] Hence, if Z_1 and Z_2 consist each of a parallel L-C combination, the inductors provide very low resistance dc paths for biasing, besides, incidentally, improving the noise performance of the oscillator. If overtone crystals are used, one of these L-C combinations must be resonant above the frequency of the next lower overtone so that its reactance at the frequency of this overtone is already positive while the reactance of the other L-C combination must still be negative. The other L-C combination must have its resonance frequency below the fundamental mode of the crystal to prevent oscillations at any one of the overtones below the desired one.

At the frequency of the desired overtone, both L-C combinations will then be capacitive. The actual values of the L's and C's necessary to meet all these requirements will generally be found to be in a reasonable range and to meet the additional requirement that no

oscillations be possible at frequencies below the lower of the two resonances, where both reactances are positive.

To avoid any oversight during these final steps, it will be found good practice to check the frequency stability of the completed design with the aid of Eq. (34).

While the questions regarding the design of the feedback network can most often be satisfactorily resolved by following the approach described, the active device will require a considerable amount of attention for high precision applications even if it will rarely pose difficulties in the design of general purpose oscillators. Of principal concern is the fact that g_m as defined in (26) usually has a phase angle θ_g that, even when it is small, is subject to change due to a large number of causes.

12. EXTENDING THE THEORY TO OTHER TYPE OSCILLATORS

Throughout the main body of this report, we have restricted the detailed discussions principally to those oscillators that have the crystal unit in the top section of the π network. Although this covers a fairly large class of circuits, it is admittedly a configuration that is most easily analyzed by the present technique. The more important circuits in this category are the Pierce oscillator and the CI meter circuit, as well as the transformer-coupled oscillator and their various modifications.

Common to all of these circuits are three and only three essential nodes in the feedback network.

More complicated feedback circuits can, as linear passive networks, always be transformed into equivalent π networks, but the equivalent impedances are generally not independent from one another and are often physically not realizable. While the conditions for oscillation can be expressed formally through the same equations which applied for the three node circuits, the interpretation is frequently so difficult that there is no practical advantage to be derived from it. Rather, such circuits are best treated by deriving the a' parameters of the feedback network in terms of the impedances Z_k as they actually appear between the various nodes. If the active element is then assumed at first to be an ideal current generator, the condition for oscillation involves only the parameter a'_{21}, i.e., the transfer admittance of the passive network and the resulting equation will have the smallest number of terms possible with this type oscillator. By proper grouping of the terms, an expression that resembles (25) can often be obtained and the developments of the three-node oscillators can be used as a guide in the analysis. The treatment of the bridged-T oscillator in Appendix C illustrates this procedure.

For transistor oscillators, the approximation of the active device by an ideal current generator is rarely adequate and additional complexities have to be expected. Nevertheless, reasonably simple relations can frequently be obtained by expanding the basic approach.

The cardinal rule here, as in treating any oscillator, is to represent the impedances between the network nodes in general terms and to introduce their resistive and reactive components only if it can no longer be avoided and only after a clear qualitative picture of their respective roles has been obtained. There is absolutely no need to use the R's, L's and C's of the actual network elements during the analysis. Within the framework of the equivalent linearization, these elements serve no other purpose in the oscillator than to physically realize the resistive and reactive components of the respective impedances, and their values can be determined accordingly at the very end of the analysis.

It will be found in practical applications that the very large number of oscillator circuits known to the art can be reduced to a few basic types whereby the major difference between oscillators within one group is in the physical realization of the network impedances, the ground connection, the position of the load, or in the active device. Nearly all of the commonly used oscillators in the HF and VHF range have a feedback circuit either in the form of a π network or a bridged-T network.

13. CONCLUSIONS

Assuming only the concept of the equivalent linearization to be valid, a rigorous and complete analysis of crystal oscillators can be carried out with comparative ease if the network impedances are represented in polar form and the geometric constructions in the impedance plane are used as a guide for the development.

The results of such an analysis make it possible to select the type of feedback network best suited for a specific application and to determine those values of the network components that will assure optimum performance.

Considering their simplicity, the π network and the Bridged-T network are by far the most important configurations for the feedback circuit in crystal oscillators and the majority of needs can undoubtedly be satisfied with either the one or the other, in conjunction with a suitable active device. Many more basic configurations, however, are possible and to provide a broader basis for the circuit selection, the pertinent relations describing the performance of at least several of them have yet to be derived and catalogued.

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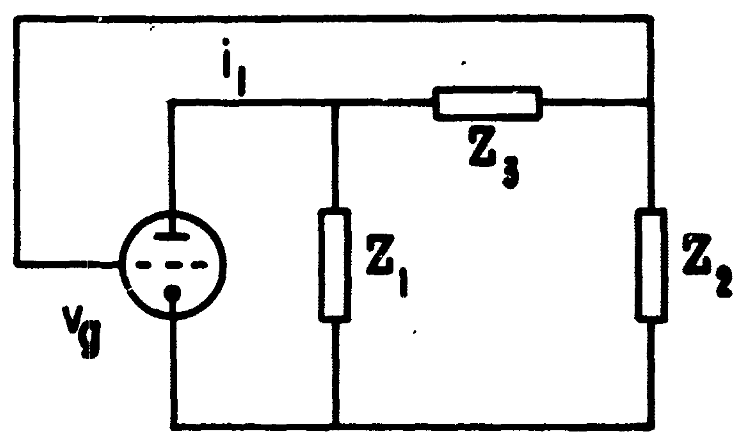


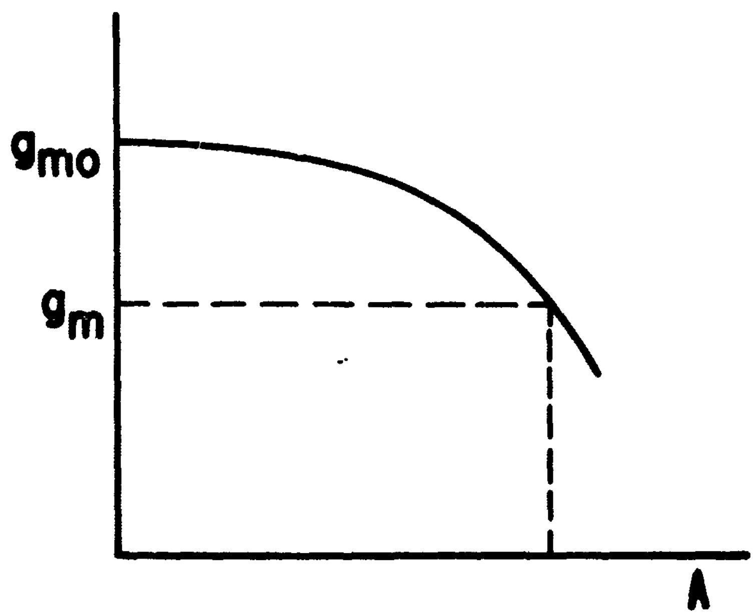
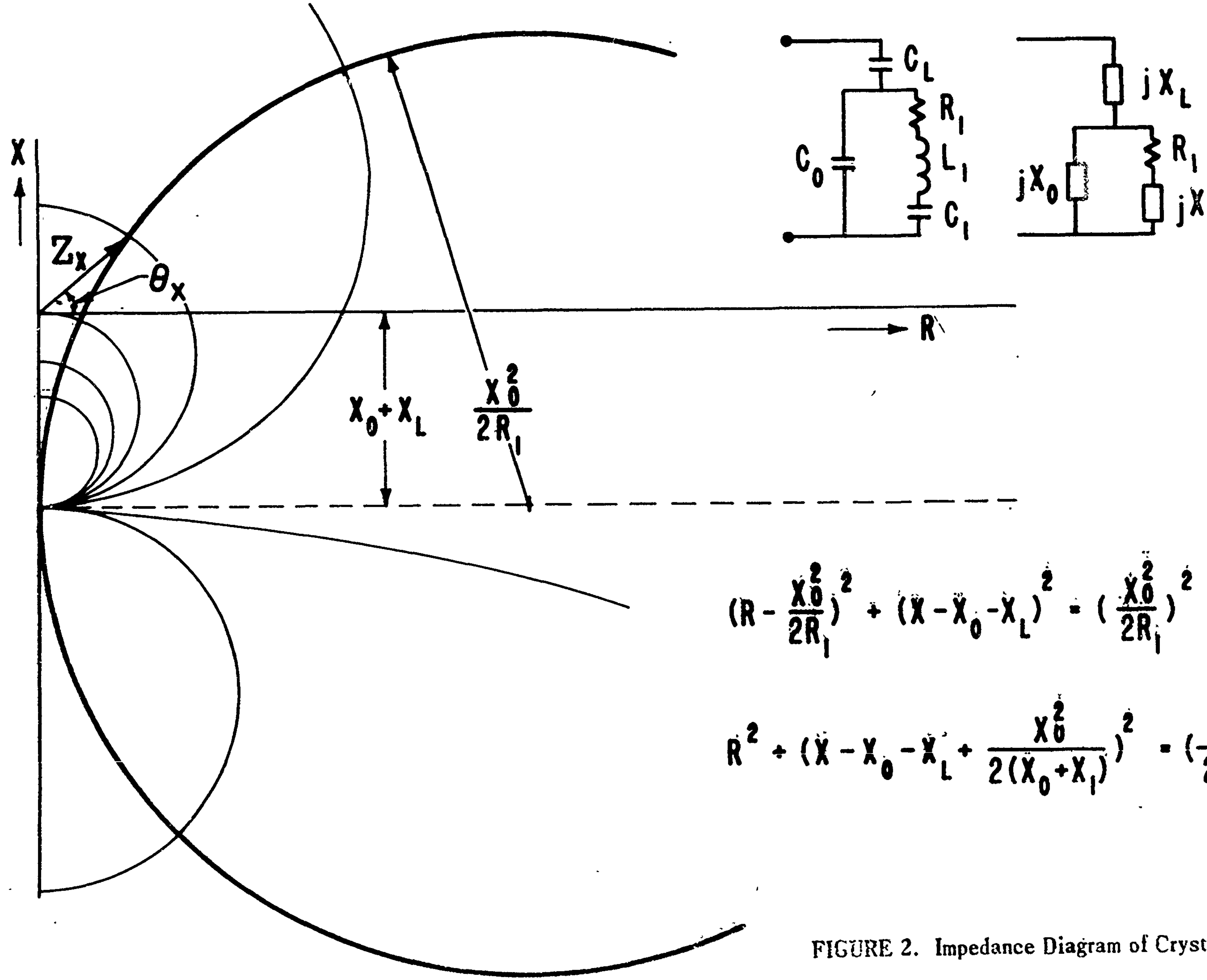


FIGURE 1. Top: Schematic of Basic Oscillator Circuit
Bottom: Typical g_m vs Amplitude Characteristic



$$\left(R-\frac{X_0^2}{2R_1}\right)^2+(X-X_0-X_L)^2=\left(\frac{X_0^2}{2R_1}\right)^2$$

$$R^2+\left(X-X_0-X_L+\frac{X_0^2}{2(X_0+X_1)}\right)^2=\left(\frac{X_0^2}{2(X_0+X_1)}\right)^2$$

FIGURE 2. Impedance Diagram of Crystal Unit

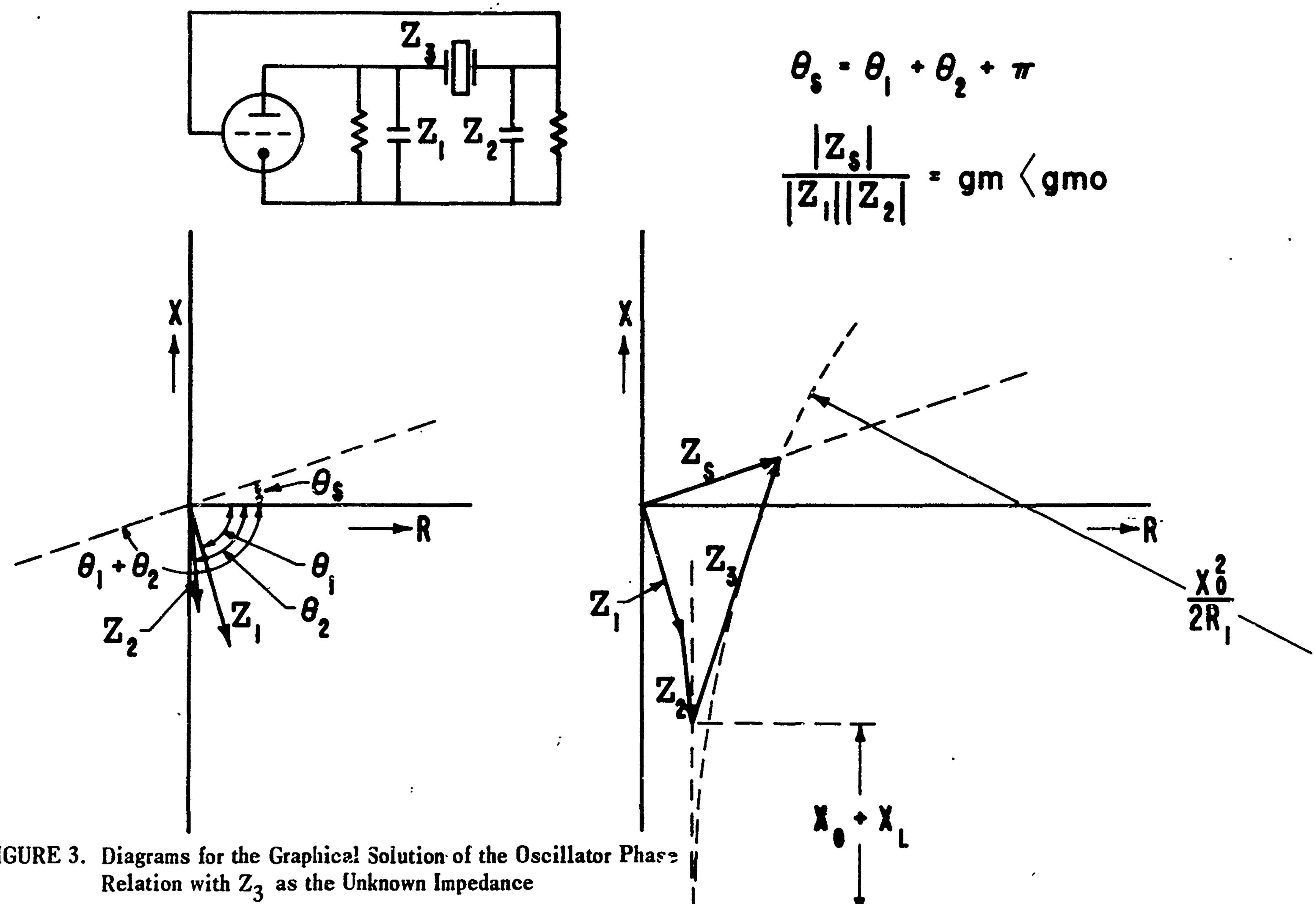


FIGURE 3. Diagrams for the Graphical Solution of the Oscillator Phase Relation with Z_3 as the Unknown Impedance

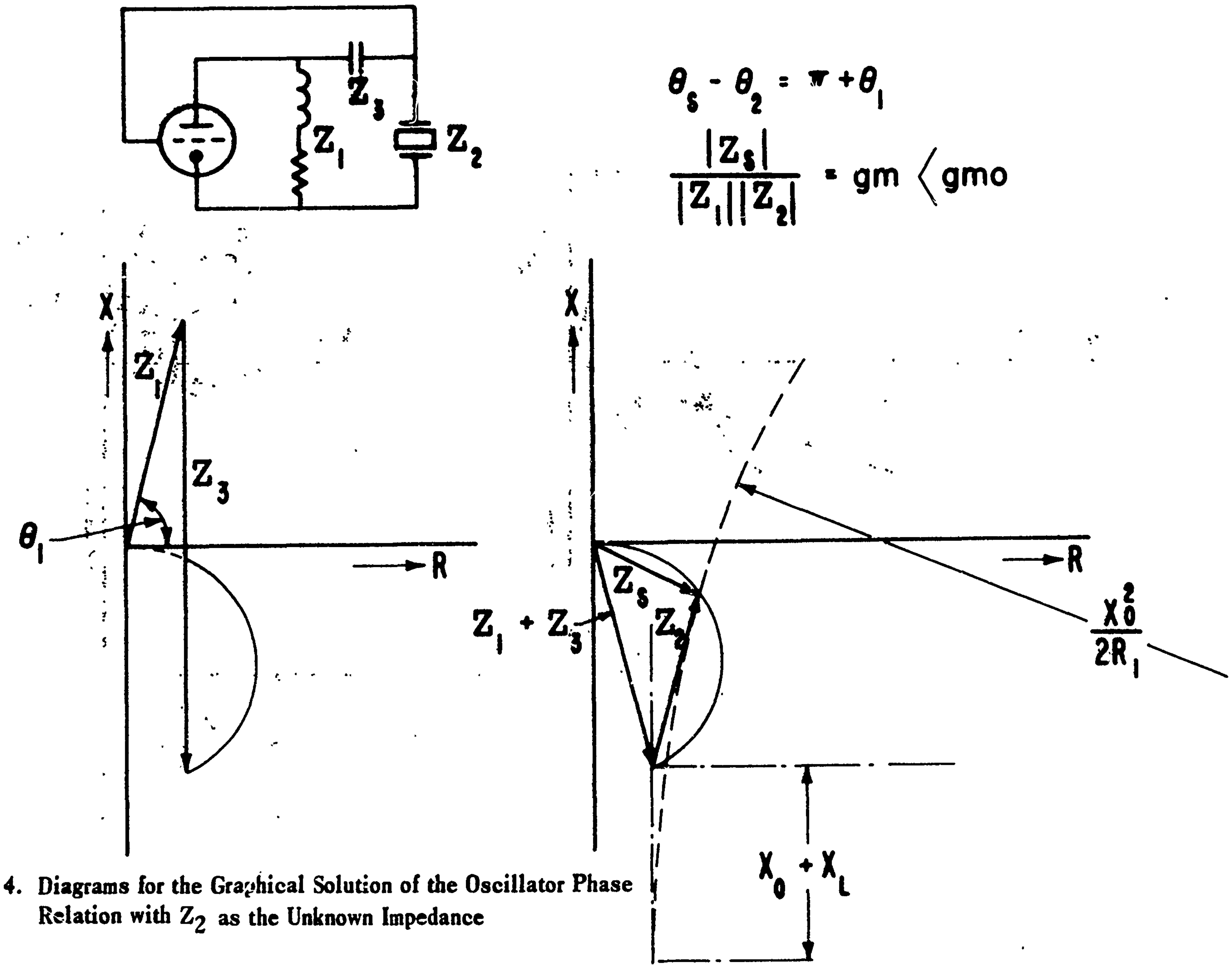
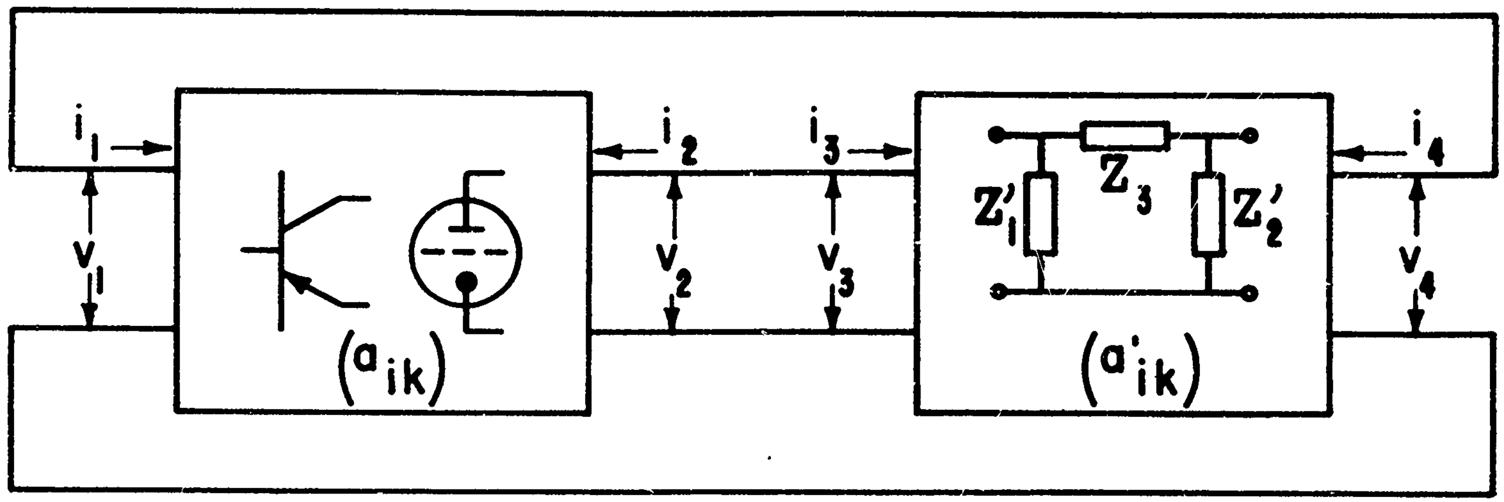


FIGURE 4. Diagrams for the Graphical Solution of the Oscillator Phase Relation with Z_2 as the Unknown Impedance



$i_1 = -i_4$

$i_3 = -i_2$

$v_1 = v_4$

$v_3 = v_2$

FIGURE 5. General Representation of a Feedback Oscillator

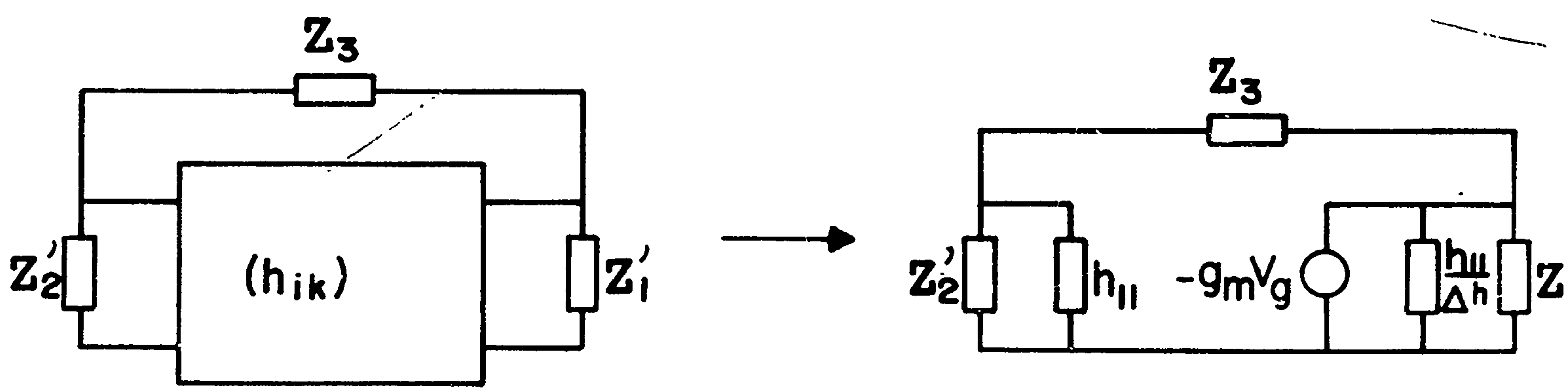


FIGURE 6. Equivalent Circuit for an Arbitrary Active Element in Oscillator Circuits

APPENDIX A:
THE EFFECTIVE TRANSCONDUCTANCE g_m

The total plate current of a vacuum tube is a function of the actual values of all the electrode potentials.[16] For the present purpose we will assume that transit time effects are negligible and that the plate resistance of the tube is infinitely large. If the control-grid voltage E_c is composed of an ac component e_g, superimposed on a dc basis voltage v_b, i.e.,

$$E_c = v_b + e_g, \tag{A-1}$$

the total plate current i_p as a function of E_c can be developed into a Taylor series:

$$i_p = f(v_b) + f'(v_b)\, e_g + 1/2!\, f''(v_b)\, e_g^{\,2} + 1/3!\, f'''(v_b)\, e_g^{\,3} + \cdots \tag{A-2}$$

This expansion holds quite generally and is shown principally to emphasize the fact that the coefficients of the various powers of e_g are functions of the bias voltage v_b and of all the other electrode potentials. For brevity we will write (A-2) as

$$i_p = i_o - \alpha e_g - \alpha_2 e_g^{\,2} + \beta e_g^{\,3} + \cdots \tag{A-3}$$

with the definitions of the symbols obvious from a comparison of the two expressions.

Figure A-1 shows two examples of curves representing plate current vs control-grid voltage as they might be obtained from static measurements. By determining the value of the function and its derivatives at the point of operation, it is possible in principle to find the values of i_o, α, α_2, β a.s.o. in (A-3) that hold for this particular value of v_b.

If an ac voltage e_g in the form

$$e_g = A \cos \omega t \tag{A-4}$$

is applied to the grid, the plate current i_p will, according to (A-3), have the form

$$i_p = i_o - \frac{\alpha_2}{2} A^2 - (\alpha - \frac{3}{4} \beta A^2) A \cos \omega t - \frac{\alpha_2}{2} A^2 \cos 2\omega t + \frac{\beta}{4} A^3 \cos 3\omega t + \cdots \tag{A-5}$$

All even powers of e_g contribute to the dc current and only the odd powers of e_g contribute to the fundamental component of the signal. It must again be emphasized that i_o, α, α_2 and β in (A-5) are functions of the bias voltage and of all the other electrode potentials. If, as is almost always the case, the bias voltage or any of the other electrode potentials are functions of the dc current through the tube, every one of the parameters in (A-3) will depend upon the amplitude of the applied signal whenever the coefficients of the even powers of e_g are not zero.[17]

In Fig. A-1(a), there is only one value for v_b, namely at the inflexion point of the curve, for which α_2 will be zero. Unfortunately, a behavior such as sketched in Fig. A-1(a) is rarely, if ever, found in actual devices. Nearly all vacuum tubes, operated normally, have a static characteristic of the type sketched in Fig. A-1(b); an evaluation of actual curves shows that for most values of v_b, β is very small compared to α_2 and can be either positive

or negative.

From (A-4) and (A-5), it is seen that the ratio of the fundamental component of the plate current to the ac component of the grid voltage, if the latter is a pure sinusoid, is given by

$$g_m = \alpha - \frac{3}{4} \beta A^2 \tag{A-6}$$

and represents the effective transconductance of the tube for the fundamental component signal, i.e., its equivalent linear transconductance.[18] According to what has just been explained, however, α and β are implicit functions of the signal amplitude that we approximate by

$$\begin{aligned} \alpha &= g_{mo} - \kappa \alpha_2 A^2 \\ \beta &= \beta_o - \kappa_2 \alpha_2 A^2 \end{aligned} \tag{A-7}$$

whereby κ and κ_2 are constants determined by the manner in which the dc electrode potentials, including v_b, depend upon the dc component of the plate current. g_{mo} and β_o are evidently the values of α and β, respectively, for infinitely small signals. For the transconductance of the tube, as defined in (A-6), we obtain the expression

$$g_m = g_{mo} - \kappa \alpha_2 A^2 - \frac{3}{4} \beta A^2 \tag{A-8}$$

which shows clearly its dependence on the amplitude of the signal.

Since κ in (A-8) is a function of the dc biasing circuit of the tube, it will obviously be extremely difficult to determine the g_m vs amplitude curve for a particular device from the static voltage current characteristics. However, it is readily possible to measure g_m and its dependence on amplitude directly by observing the fundamental component of the plate current in response to a sinusoidal input voltage of known amplitude. Rather simple techniques are available to carry out such measurements directly.[19]

An extension of the considerations given here to transistors will show that completely analogous situations exist in all active devices.

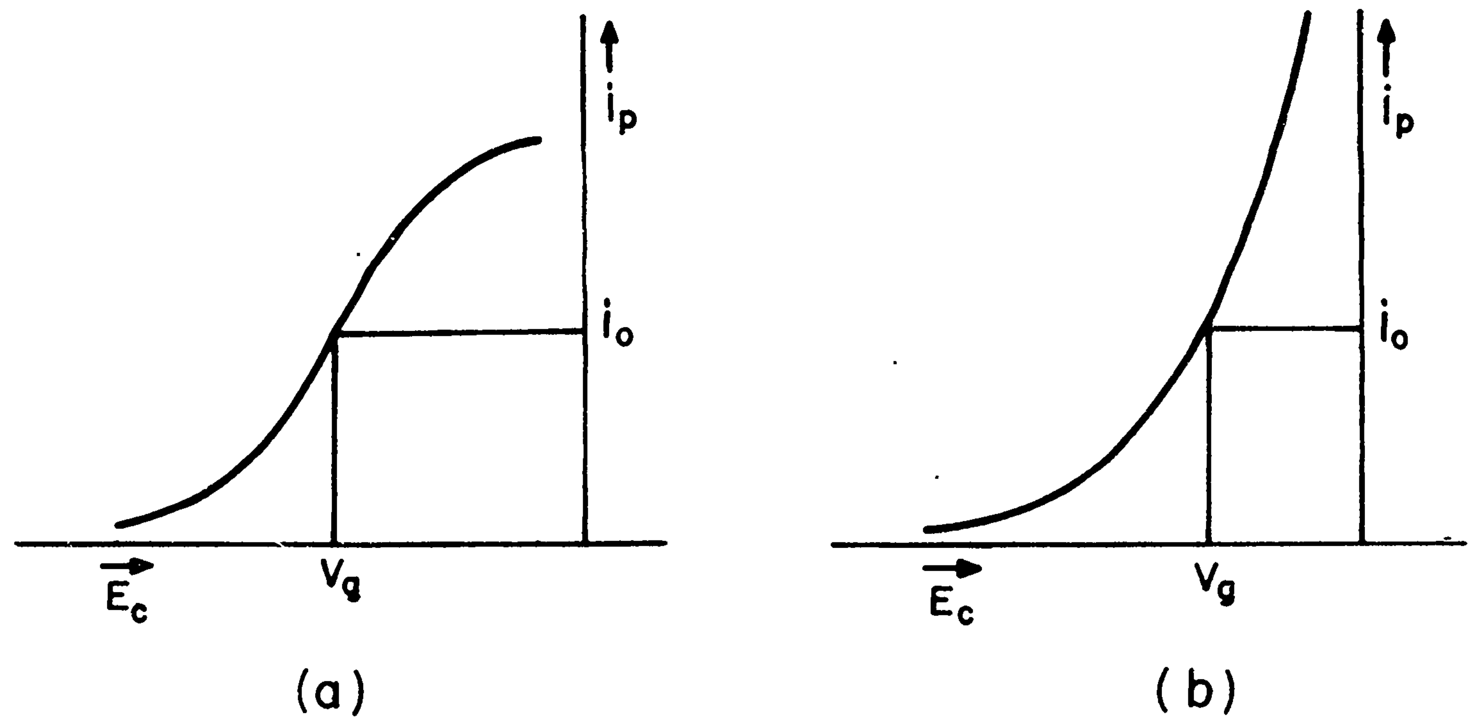


FIGURE A-1. Representative Plate Current vs Control Grid Voltage Characteristics of Vacuum Tubes

APPENDIX B:

THE NONLINEAR OSCILLATOR

To clarify the influence of the nonlinearities in the active device and their equivalent linear representation on oscillator behavior, we consider the circuit shown in Fig. B-1. This oscillator, though still readily tractable analytically to the extent intended, has sufficient similarity to a quartz-crystal oscillator to provide a valid description of the major performance aspects of the latter.

The transformer in Fig. B-1 has a windings ratio of 1:1 and unity coupling. It presents an inductive reactance ωL_1 to the circuit and provides the 180° phase shift required by this configuration.

The differential equation for the ac grid voltage e_g is

$$\ddot{e}_g + \frac{R_3}{L}\dot{e}_g + \omega_o^2 e_g = \frac{L_1}{LC_2}\frac{d}{dt} i_p . \tag{B-1}$$

whereby

$$L = L_1 + L_3$$
$$\frac{1}{C} = \frac{1}{C_2} + \frac{1}{C_3} \tag{B-2}$$
$$\omega_o^2 = \frac{1}{LC} .$$

Since the total plate current i_p is a function of e_g, such as dealt with in Appendix A, a rigorous treatment of equation (B-1) would require to take note of the fact that i_o, α, α_2 and β in (A-3) depend upon the amplitude of e_g and hence are also functions of e_g. Such a treatment, which definitely should be carried out to properly describe the initial transient behavior of an oscillator, has apparently never been reported and in fact will not be required for the present purpose. We are interested mainly in the steady-state solution of (B-1) and therefore are justified in assuming i_o, α, α_2 and β to be constants.

During steady state, e_g will be, if it is different from zero, a periodic function of time that can be developed into a Fourier series. Its shape can be determined with arbitrary precision by substituting this Fourier series into (B-1), together with (A-3), and solving the system of equations that results from applying the principle of the harmonic balance.[20] However, if the harmonic content in the plate current is very small or if the feedback network in Fig. B-1 is such as to prevent these harmonic components to reach the grid, or both, e_g will be very nearly a pure sinusoid and we can write

$$e_g = e_{go} + O(n\omega) \tag{B-3}$$

whereby $O(n\omega)$ are harmonic terms small of higher order and

$$e_{go} = A \cos \omega t \tag{B-4}$$

It can then be shown by combining (B-3), (B-1) and (A-5) that e_{go} is the solution of the following differential equation:

$$\ddot{e}_{go} + \frac{L_1}{LC_2}\left(\frac{C_2R_3}{L_1} - \alpha + \frac{3}{4}\beta A^2\right)\dot{e}_{go} + \omega_o^2 e_{go} = 0 \tag{B-5}$$

which can alternatively be written with (A-6) as

$$\ddot{e}_{go} + \frac{L_1}{LC_2}\left(\frac{C_2R_3}{L_1} - g_m\right)\dot{e}_{go} + \omega_o^2 e_{go} = 0 \; . \tag{B-6}$$

While this equation was derived here mainly to describe the first approximation to the steady-state solution of (B-1), it is general enough to illustrate qualitatively a number of important aspects of oscillator behavior. In fact, the solution of (B-6)

$$e_{go} = A\,e^{-\gamma t}\cos\omega_o t \tag{B-7}$$

with

$$\gamma = \frac{L_1}{C_2}\left(\frac{C_2R_3}{L_1} - g_m\right) \tag{B-8}$$

is an excellent piecewise approximation to the actual behavior of the grid voltage if the change in amplitude with time is very slow compared to the period of the oscillations.[21] Obviously, steady-state oscillations require that

$$\frac{C_2R_3}{L_1} - g_m = 0 \; . \tag{B-9}$$

It will be observed that this condition is identical to the condition (6) in Section 2, as, of course, it must be since

$$\frac{C_2R_3}{L_1} = \frac{|Z_s|}{|Z_1||Z_2|} \tag{B-10}$$

as seen from Fig. B-1. The phase relation, incidentally, for this circuit is $\theta_1 + \theta_2 - \theta_s = 0$ rather than $= \pm\pi$ because of the 180° phase shift due to the transformer.

Since g_m depends upon the amplitude of the oscillations, the steady-state amplitude can be evaluated from (B-9) if this dependence is known. It is clear from (B-7) that the amplitude of oscillations will increase if g_m is larger than C_2R_3/L_1 and will decrease if it is smaller than this quantity. For the oscillator to function properly, it is required that g_m decrease as the amplitude increases which, in turn, requires that κ and α_2 in (A-8) have the same sign and that β be negative. The latter is not always the case in vacuum tubes and depending upon the relative magnitudes of the two terms, the g_m vs amplitude curve occasionally rises at first until it is forced down by the higher order terms in A as α_2 and β are again functions of amplitude. Actual measurements, however, show that g_m does decrease monotonically with amplitude in the large majority of practical cases.

It is seen further from (B-6) that C_2R_3/L_1 must be smaller than g_{mo}, the transconductance of the device for extremely small signals, for oscillations to start from noise impulses, i.e., from extremely small amplitudes. The rate of increase of the oscillations is strongly dependent upon $L_1/(L_1 + L_3)$ while the final amplitude is independent of this ratio. Hence, if the R_3 L_3 C_3 branch in Fig. B-1 represents a crystal unit resonant at ω_o and L_1 and C_2 are adjusted such that $\omega^2 = 1/L_1C_2$, the same amplitude of steady-state oscillations will be reached if this crystal is replaced by a resistor of value R_3. This fact is, of course, the basis for the substitution method for determining the crystal resistance.[22] The time required to reach the final amplitude, however, is shorter with the resistor by the ratio of the respective circuit Q's.

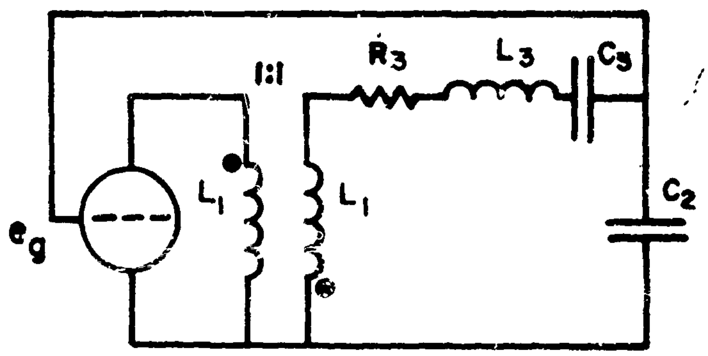


FIGURE B-1. Skeleton Model of a Crystal Oscillator

APPENDIX C:

THE IDEALIZED BRIDGED-T OSCILLATOR

A bridged-T oscillator with an ideal current generator as the active element is shown in Fig. C-1. According to the comments in Section 12, it is best in this case to use the general condition for oscillation (22) that now reduces to

$$1 - a_{12} a'_{21} = 0 \tag{C-1}$$

since only a_{12} is different from zero for the ideal current generator. a'_{21} can readily be found[23] for the bridged-T network and, after some rearrangement of terms, the condition for oscillations is obtained in the form

$$\frac{Z_s}{Z_1 Z_2}\left(\frac{1}{g_m} + Z_4\right) + 1 = 0 \tag{C-2}$$

with

$$Z_s = Z_1 + Z_2 + Z_3 \ .$$

If we introduce an auxiliary vector

$$Z_A = \frac{1}{g_m} + Z_4 \tag{C-3}$$

much of the same techniques used previously can again be applied to analyze the performance of this oscillator. We will restrict our comments here, however, to only a qualitative description of some of the more important aspects of its behavior.

Separating the real and imaginary parts of (C-2) gives, with (C-3) the following conditions

$$\frac{|Z_s|}{|Z_1|}\frac{|Z_A|}{|Z_2|} = 1 \tag{C-4}$$

$$\theta_A + \theta_s = \theta_1 + \theta_2 + \pi \ . \tag{C-5}$$

As the simplest representative of this type oscillator, we consider the circuit shown in Fig. C-2. The impedance Z_3 of the parallel circuit formed by R_L, C_3, L_3 is most conveniently expressed in the form

$$Z_3 = R_3 + j\,X_3 = \frac{R_L}{1+\rho^2} + j\,\frac{R_L \rho}{1+\rho^2} \tag{C-6}$$

with

$$\rho = Q_3 \left(1 - \frac{\omega^2}{\omega_3{}^2}\right) \quad ; \quad Q_3 = \frac{R_L}{\omega L_3} \tag{C-6}$$

$$\omega_3{}^2 = \frac{1}{L_3 C_3} \quad .$$

In this circuit $\theta_1 = \theta_2 = -\pi/2$ and the phase relation (C-5) reduces to

$$\theta_A + \theta_S = 0 \quad . \tag{C-7}$$

If the crystal unit should be operated at series resonance and g_m has zero phase angle, θ_A will be zero and the condition for oscillation requires θ_S to be zero also, i.e., the circuit formed by C_1, C_2 and Z_3 must be at resonance:

$$X_1 + X_2 + X_3 = 0 \quad . \tag{C-8}$$

The reactances of C_1 and C_2 must satisfy in this case the condition

$$\frac{|X_1|\,|X_2|}{R_3} = \left(\frac{1}{g_m} + R_4\right) > \left(\frac{1}{g_{mo}} + R_4\right) \tag{C-9}$$

for steady-state oscillations to be possible. R_4 thereby is the resonance resistance of the crystal unit.

From (C-6), it is observed that X_3 and R_3 are related by

$$X_3 = \rho R_3 \quad . \tag{C-10}$$

The conditions for oscillation (C-8) and (C-9) can be solved for X_1 and X_2, and one finds for the circuit in Fig. C-2

$$X_1 = -\frac{1}{\omega C_1} = -\frac{\rho}{2} R_3 (1 + q) \tag{C-11}$$

$$X_2 = -\frac{1}{\omega C_2} = -\frac{\rho}{2} R_3 (1 - q) \quad .$$

$$q = \sqrt{1 - \frac{4}{\rho^2 R_3} \left(\frac{1}{g_m} + R_4\right)} \quad . \tag{C-12}$$

Whenever the relations (C-11) are satisfied, oscillations will take place. Aside from the fact that q must be real, there are no restrictions on ρ and R_3. Hence, the conditions for oscillation yield only two relations between the seven oscillator parameters C_1, C_2, L_3, C_3, R_L, R_4 and g_m. They can evidently be satisfied over a very wide range for either of these parameters. This again emphasizes the well-known truth that it is not at all difficult to assemble a configuration that will oscillate somehow.

Since, however, the conditions for oscillation above leave the problem of oscillator design vastly underdetermined, additional relations between the parameters can be chosen to satisfy additional requirements that might be placed on the oscillator.

As a specific example, it will be assumed that the principal design objectives are high-frequency stability and high output power for a given crystal power. The additional relations between the circuit parameters have to be formulated such that those requirements are met. This will be illustrated in the following paragraphs.

The ratio of the power dissipated in Z_3 in Fig. C-1 to that dissipated in Z_4 is, quite generally, given by

$$\frac{P_3}{P_4} = \frac{|Z_1|^2}{|Z_S|^2} \frac{\text{Real } Z_3}{\text{Real } Z_4} . \tag{C-13}$$

If the dominant resistive element in Z_3 is due to the load, and in Z_4 is due to the crystal unit, (C-13) can be used immediately to determine the output power to crystal power ratio. The absolute power dissipated in either the load or the crystal unit depends upon the difference between g_m and g_{mo} and is determined by the amplitude dependence of g_m.

Considering (C-8), (C-9) and (C-11), together with the fact that Z_1 and Z_2 in Fig. C-2 are purely reactive, the ratio (C-13) assumes in the present case the form

$$\frac{P_3}{P_4} = \frac{\frac{1}{g_m} + R_4}{R_4} \frac{1 + q}{1 - q} . \tag{C-14}$$

If g_m and R_4 are given quantities, it is evidently desirable that q be as close to one as practical, in order to obtain a large power ratio. This establishes the first of the additional conditions.

The stability relation in general form is found from (C-2) through differentiation as

$$\frac{\Delta Z_1 + \Delta Z_2 + \Delta Z_3}{Z_S} - \frac{\Delta Z_1}{Z_1} - \frac{\Delta Z_2}{Z_2} + \frac{\Delta Z_A}{Z_A} = 0 \cdot \tag{C-15}$$

Its imaginary part contains all the information required to determine the relative change in the frequency of oscillation, $\Delta\omega/\omega$, in response to small variations in the circuit elements. A procedure very similar to that used in Sections 6 and 8 can be followed to bring the resulting expression into a convenient and instructive form. The fact that a specific circuit configuration, namely that in Fig. C-2, is to be dealt with now can be taken advantage of to simplify some of the steps. In the following, each of the terms in (C-15) will first be discussed individually, and then their imaginary parts will be collected.

The impedance Z_3 is defined by (C-6). As a vector in the impedance plane, $Z_3 = Z_3(\omega)$ follows the circle shown in Fig. C-3 which is described by the equation

$$\left(R - \frac{R_L}{2}\right)^2 + X^2 = \left(\frac{R_L}{2}\right)^2 \cdot \tag{C-16}$$

This can readily be verified by eliminating the frequency variable from R_3 and X_3 in (C-6).

From Fig. C-2, it is apparent that Z_3 is a function of C_3, L_3, R_L and ω. According to the relation (12) in Section 6, Z_3 has four components, which are found from (C-6) by differentiation

$$\frac{\partial Z_3}{\partial L_3} \Delta L_3 = R_3 Q_3 \frac{\Delta L_3}{L_3} e^{j\theta_t} \tag{C-17}$$

$$\frac{\partial Z_3}{\partial C_3} \Delta C_3 = R_3 Q_3 \frac{\omega^2}{\omega_3^2} \frac{\Delta C_3}{C_2} e^{j\theta_t} \tag{C-18}$$

$$\frac{\partial Z_3}{\partial R_L} \Delta R_L = R_3 \frac{\Delta R_L}{R_L} e^{j\theta_n} \tag{C-19}$$

$$\frac{\partial Z_3}{\partial \omega} \Delta\omega = R_3 Q_3 \left(1 + \frac{\omega^2}{\omega_3^2}\right) \frac{\Delta\omega}{\omega} e^{j\theta_t} \tag{C-20}$$

whereby

$$\tan \theta_t = -\frac{1-\rho^2}{2\rho} \tag{C-21}$$

$$\tan \theta_n = \frac{2\rho}{1-\rho^2} . \tag{C-22}$$

The phase angles θ_t and θ_n are, respectively, the phase angles of the tangent and the normal to the impedance circle for Z_3, as indicated in Fig. C-3.

The expression (C-20), it will be noted, leads to formula (16) in Section 6, where $\omega^2 \approx \omega_3^2$ is assumed.

The impedance Z_s is found, as illustrated in Fig. C-4, by adding vectorially Z_1, Z_2 and Z_3. Evidently, since $\theta_1 = \theta_2 = -\frac{\pi}{2}$ and $\theta_s = 0$ is assumed in the present case, it follows that

$$Z_s = R_3 \tag{C-23}$$

when the condition for oscillation (C-8) is fulfilled.

With Z_1 and Z_2 purely capacitive as in Fig. C-2, the terms in (C-15) containing ΔZ_1 and ΔZ_2 present no difficulties. $\Delta Z_1/Z_1$ and $\Delta Z_2/Z_2$ are both real and give no contribution to the imaginary part of (C-15). $\Delta Z_1/Z_s$ and $\Delta Z_2/Z_s$ are, because of (C-23), both imaginary and with (C-11) can be written as

$$\frac{\Delta Z_1}{Z_s} = j \frac{\rho}{2} (1 + q) \left(\frac{\Delta\omega}{\omega} + \frac{\Delta C_1}{C_1}\right) \tag{C-24}$$

$$\frac{\Delta Z_2}{Z_s} = j\,\frac{\rho}{2}\,(1 - q)\left(\frac{\Delta\omega}{\omega} + \frac{\Delta C_2}{C_2}\right). \tag{C-25}$$

The impedance Z_A is defined by (C-3). Its construction in the R-X plane is illustrated in Fig. C-5. Evidently, Z_A follows the impedance circle of the crystal unit, translated by $\frac{1}{g_m}$. The change in Z_A with frequency is solely due to the change in crystal impedance with frequency and one finds

$$\Delta Z_A = \frac{\partial Z_4}{\partial\omega}\,\Delta\omega = 2R_4\,Q_o\,\frac{\Delta\omega}{\omega}\,e^{j\theta_{\Delta Z_4}} \tag{C-26}$$

whereby $\theta_{\Delta Z4}$ is equal to the phase angle of the tangent to the crystal impedance circle at the operating frequency.

If the crystal unit is operated at series resonance and g_m is real as assumed in the present case, θ_A equals zero. The relative change in Z_A with frequency becomes

$$\frac{\Delta Z_A}{Z_A} = \frac{2\Delta\omega}{\omega}\,Q\text{ eff}\,e^{j\theta_{\Delta Z_4}} \tag{C-27}$$

with

$$Q\text{ eff} = \frac{R_4}{\frac{1}{g_m} + R_4}\,Q_o\,. \tag{C-28}$$

All the individual terms in (C-15) are now known and their imaginary parts can be collected to determine the frequency change $\frac{\Delta\omega}{\omega}$ as a function of small variations in the circuit parameters. With proper grouping, one finds

$$\frac{\Delta\omega}{\omega}\left(\rho + Q_3\left(1 + \frac{\omega^2}{\omega_3^{\,2}}\right)\sin\theta_t + 2\,Q_{eff}\sin\theta_{\Delta Z_4}\right) + \tag{C-29}$$

$$+\frac{\rho}{2}\left((1+q)\frac{\Delta C_1}{C_1} + (1-q)\frac{\Delta C_2}{C_2}\right) + Q_3\left(\frac{\Delta L_3}{L_3} + \frac{\omega^2}{\omega_3^{\,2}}\frac{\Delta C_3}{C_3}\right)\sin\theta_t + \frac{\Delta R_L}{R_L}\sin\theta_n = 0\,.$$

To obtain the highest stability requires obviously that the coefficient of $\frac{\Delta\omega}{\omega}$ be as large as possible and the coefficients of the circuit parameter variations be as small as possible.

In crystal oscillators the dominant term in the coefficient of $\frac{\Delta\omega}{\omega}$ is ($2\,Q_{eff}\sin\theta_{\Delta Z4}$). To bring $\theta_{\Delta Z4}$ close to 90°, it may be necessary, particularly at VHF frequencies, to resonate the capacitance C_o of the crystal unit with a parallel inductor. The effective quality factor Q_{eff} depends, according to (C-28), only upon R_4 and g_m. Unlike the situation in the Pierce oscillator, it cannot be improved by appropriate choice of the elements in the

feedback network.

The values of the network parameters are, nevertheless, of considerable importance in determining the frequency stability of the oscillator. The coefficients of the circuit parameter variations in (C-29) are controlled mainly by ρ and Q_3. This becomes even more evident if one notes that

$$\sin \theta_t = \frac{\rho^2 - 1}{\rho^2 + 1}$$

$$\sin \theta_n = \frac{2\rho}{\rho^2 + 1} \tag{C-30}$$

which follows from (C-21) and (C-22). The influence of q on (C-29) is weak because $0 < q < 1$, as seen from (C-12).

The stability relation (C-29) and the expression (C-14) for the power ratio contain three parameters, namely ρ, Z_3 and q, which can be assigned specific values such that the performance of the oscillator has the desired characteristics. The expressions for ρ and Q_3 in (C-6) and the expression (C-12) for q, together with the conditions for oscillation (C-11), provide five equations to uniquely determine the values of the five circuit elements C_1, C_2, L_3, C_3, and R_L in Fig. C-2. While the values for ρ, Q_3, q, g_m and R_4 can, in principle, be selected independently from one another, practical considerations will nearly always limit their range.

Assuming that all the parameter variations in (C-29) are unrelated to one another and that their relative magnitudes $\Delta C_1/C_1$, $\Delta C_2/C_2$ a.s.o. are about equal, it can be seen that $\Delta\omega/\omega$ will be smallest when ρ is near unity. In fact, for $\rho = 1$ the frequency is independent of ΔC_3 and ΔL_3. The precise value of ρ becomes less critical for low values of Q_3.

It is concluded that high stability of the oscillator frequency and a high power ratio require $\rho = 1$, a low value of Q_3 and, because of (C-14), a q close to unity.

According to (C-6), $R_3 = R_L/2$ for $\rho = 1$, and the equations for the values of the circuit elements in Fig. C-2 can be written as follows

$$R_L = \frac{8}{1 - q^2}\left(\frac{1}{g_m} + R_4\right)$$

$$\omega L_3 = \frac{R_L}{Q_3}$$

$$\omega C_3 = \frac{1}{\omega L_3}\,\frac{Q_3 - 1}{Q_3} \tag{C-31}$$

$$\omega C_1 = \frac{1}{R_L(1 + q)}$$

$$\omega C_2 = \frac{1}{R_L(1 - q)} .$$

In order to reduce the effects of stray elements, it is generally necessary to avoid operation at very high impedance levels. With g_m and R_4 given it will frequently be found advisable, therefore, to start the design by choosing a reasonable value for ωL_3 and to determine R_L such as to arrive at an acceptable compromise between the demands for low Q_3 and a small value for (1–q).

The equations (C-31) will yield the proper values for the network elements. However, final adjustments are always best carried out on the oscillator itself. The most suitable parameters to be used are L_3 and/or C_3 as indicated by the stability relation (C-29). This fact can be illustrated with the aid of Fig. C-3.

The impedance $Z_3(\omega)$ of the R_L, C_3, L_3 circuit in Fig. C-2 moves in a clockwise direction around the circle in Fig. C-3 as the resonance frequency ω_3 of this circuit is lowered by increasing L_3 or C_3 or both, while R_L remains constant. As $Z_3(\omega)$ passes thereby the point (1) on the circle, the phase angle θ_t goes through zero and, consequently, sin θ_t in (C-29) changes its sign. The frequency of oscillation will go through a minimum at this point. Since the condition $\rho = 1$ requires that the R_L, C_3, L_3 circuit be operated at the point (1), this frequency minimum provides a convenient criterion for the fine tuning of C_3 and/or L_3. Retuning of C_1 and/or C_2 is required if the minimum frequency does not occur at the resonance frequency of the crystal unit.

It is apparent that the tuning of the oscillator becomes rather critical if enough performance requirements are imposed to completely specify the oscillator circuit, as is the case in the example just given. Whatever the application, however, the relations (C-14) and (C-29) will always be found useful to arrive at an acceptable compromise in the design.

It must be emphasized that all relations in Appendix C were derived under the assumption that the active device is an ideal current generator. The extension of this development to transistor oscillators will be reported elsewhere.

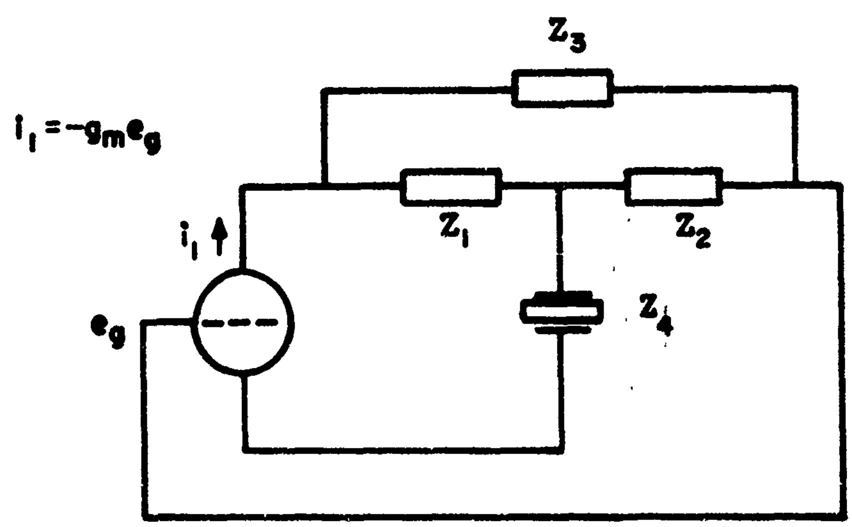


FIGURE C–1. Idealized Bridged–T Oscillator

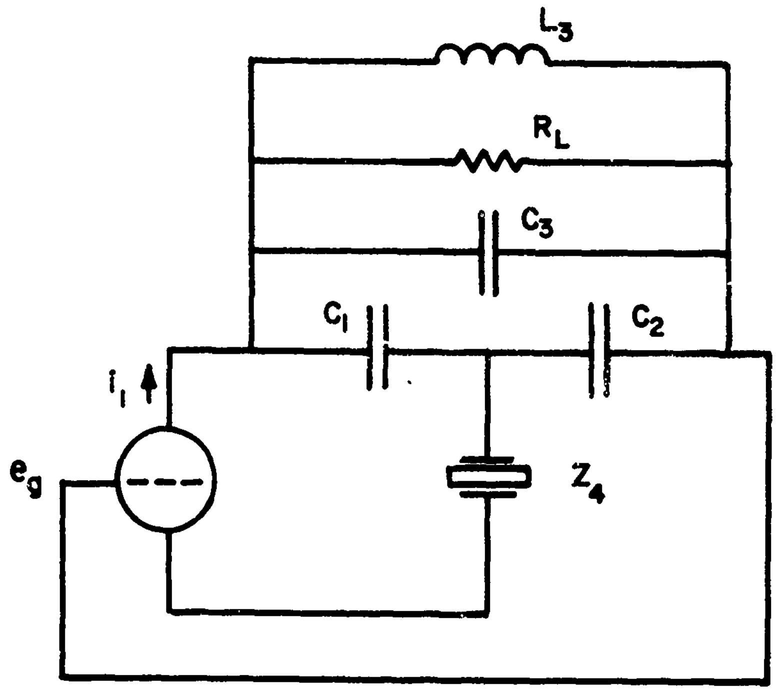


FIGURE C–2. Typical Configuration of Bridged–T Oscillator Network

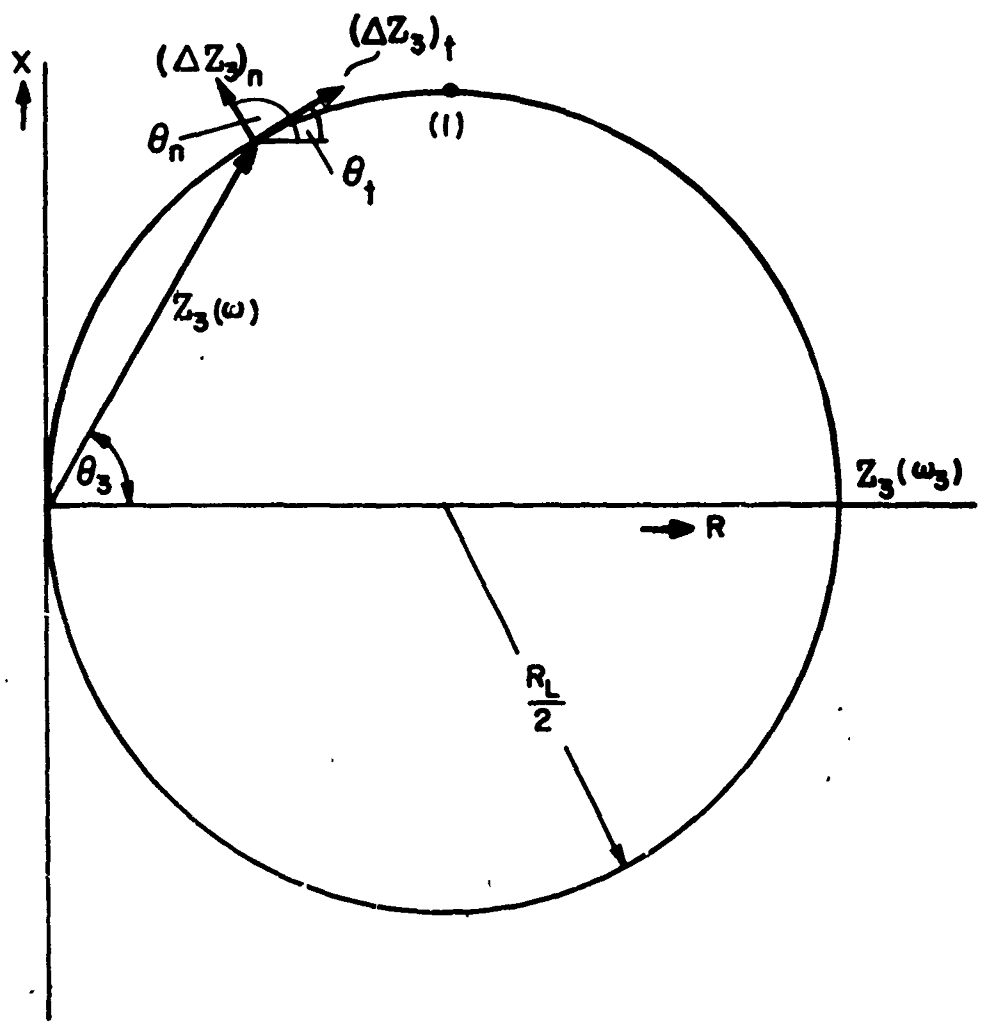


FIGURE C–3. Impedance Diagram of R_L–C_3–L_3 Parallel Resonance Circuit

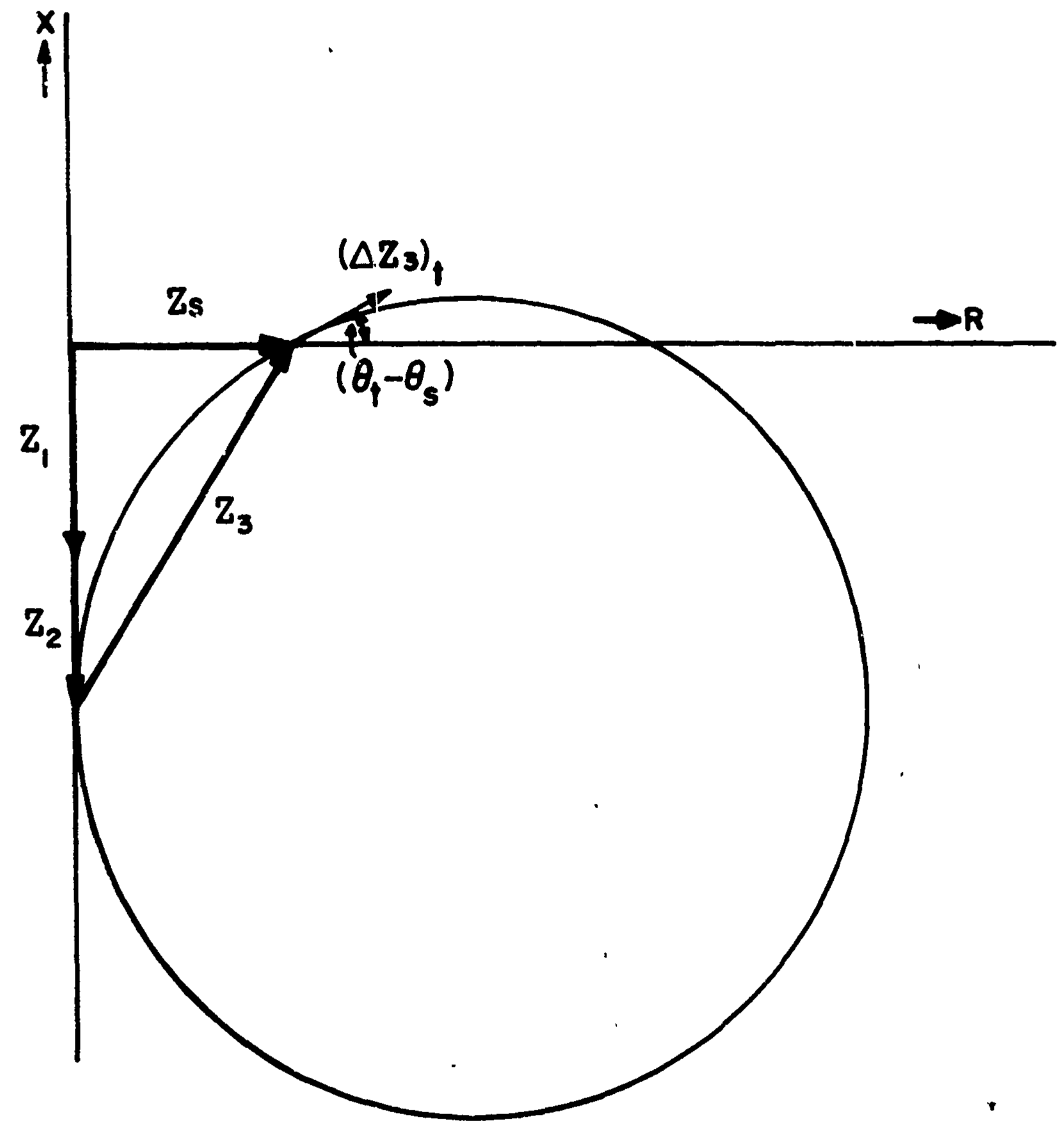


FIGURE C–4. Diagram Illustrating Graphical Construction of $Z_s = Z_1 + Z_2 + Z_3$.

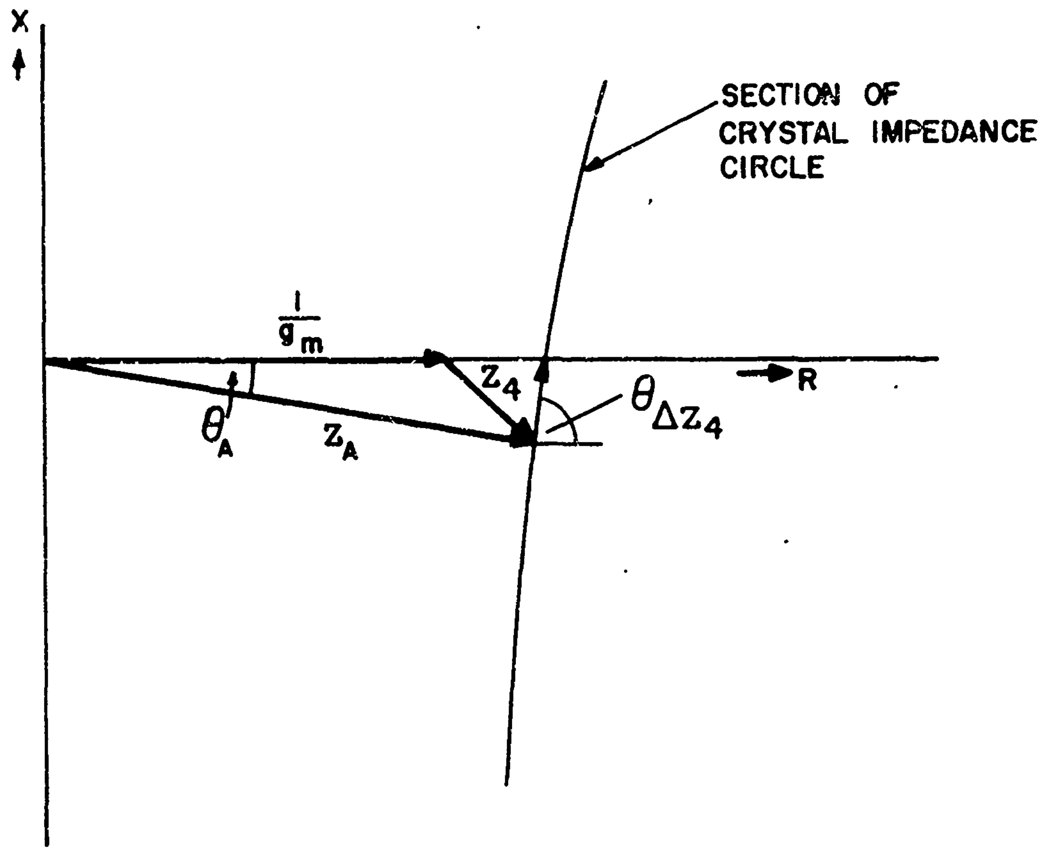


FIGURE C–5. Diagram Illustrating Graphical Construction of Impedance Z_4 Defined by Equation (C–3).

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13. ABSTRACT

The approach developed in this report appears to satisfy all the major requirements that must be placed on a unifying technique for oscillator analysis and design. As demonstrated on specific examples, the conditions for oscillation in a generally valid form can be processed to determine (a) the amplitude of oscillation in relation to the characteristics of the active device, (b) the requirements on the feedback network to operate the crystal unit according to its specifications, (c) the changes in frequency of oscillation in response to variations in any one of the circuit components and, hence, the values required for these components to obtain maximum stability, and (d) the output power in proportion to the power dissipated in the crystal unit.

The key to this approach lies in a graphical method for solution of the oscillator phase equation in the impedance plane. The impedance diagrams obtained thereby open the way to a thorough qualitative understanding of the cause-effect relationships in oscillator performance and provide the guidelines to bring the analytic expressions into a convenient form for quantitative work. Detailed discussions are carried out for the Pierce oscillator and the bridged-T oscillator to illustrate the practical application of the approach. (Author)

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Vacuum Tube						
Transistor						
Quartz Crystal Unit						
Optimum Design						
Stability Relations						
Impedance Plane						
Graphical Solution of Phase Relation						
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